1. Warmup: Permanent Income Hypothesis

Solve the deterministic individual problem

$$\max_{\{c_t,a_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{s.t. } c_t + a_{t+1} = y_t + (1+r)a_t \quad \forall t$$

f.o.c.'s are

$$c_t: \quad u'(c_t) = \lambda_t$$
$$a_{t+1}: \quad \lambda_t = \beta(1+r)\lambda_{t+1}$$

Assuming $\beta(1+r) = 1$, we obtain

$$u'(c_t) = u'(c_{t+1}).$$

Hence if $u(\cdot)$ is strictly concave, $c_{t+j} = c_t$ for all j. Then writing out the budget constraints,

$$\begin{aligned}
c_t + a_{t+1} &= y_t + (1+r)a_t \\
c_t + a_{t+2} &= y_{t+1} + (1+r)a_{t+1} \\
c_t + a_{t+3} &= y_{t+2} + (1+r)a_{t+2} \\
c_t + a_{t+4} &= y_{t+3} + (1+r)a_{t+3} \\
&\cdots
\end{aligned}$$

Multiply iteratively by $\frac{1}{1+r}$ to get

$$\begin{aligned}
c_t + a_{t+1} &= y_t + (1+r)a_t \\
\frac{1}{1+r}c_t + \frac{1}{1+r}a_{t+2} &= \frac{1}{1+r}y_{t+1} + a_{t+1} \\
\left(\frac{1}{1+r}\right)^2 c_t + \left(\frac{1}{1+r}\right)^2 a_{t+3} &= \left(\frac{1}{1+r}\right)^2 y_{t+2} + \left(\frac{1}{1+r}\right) a_{t+2}
\end{aligned}$$

$$\left(\frac{1}{1+r}\right)^3 c_t + \left(\frac{1}{1+r}\right)^3 a_{t+4} = \left(\frac{1}{1+r}\right)^3 y_{t+3} + \left(\frac{1}{1+r}\right)^2 a_{t+3}$$

$$\cdots$$

Then adding up LHS's and RHS's, notice that the a's cancel out, so

$$\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j c_t + \lim_{J\to\infty}\left(\frac{1}{1+r}\right)^J a_{t+J+1} = \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j y_{t+j} + (1+r)a_t$$

By the TVC or borrowing constraint, the last term of LHS is 0. Hence

$$\frac{1+r}{r}c_t = \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j y_{t+j} + (1+r)a_t$$

So

$$c_t = \frac{r}{1+r}\cdot\left\{y_t + \sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^j y_{t+j} + (1+r)a_t\right\}.$$

Friedman's conjecture is that this holds even in the stochastic case:

$$c_t = \frac{r}{1+r}\cdot\left\{y_t + \mathbb{E}_t\sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^j y_{t+j} + (1+r)a_t\right\}. \tag{1}$$

This is called "certainty equivalence," a principle that is exploited in many other applications as well to show that the solution to a stochastic problem coincides with its deterministic counterpart. Of course, it is something that has to be shown case by case, not a universal property.

2. The Income Fluctuation Problem

This is a summary of some important propositions from Huggett (1993); Aiyagari (1993); Chamberlain and Wilson (2000).

2.1 $\beta(1+r) < 1$ w/o uncertainty

Assume a deterministic income process $\{y_t\}_{t=0}^{\infty}$ s.t. $y_t = \bar{y}$ for all t, i.e., the endowment is same every period. The individual's problem is

$$
\begin{aligned}
&\max_{\{c_t, a_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t), \\
&\text{s.t.} \quad c_t + a_{t+1} = \bar{y} + (1+r)a_t \\
&\qquad\quad a_{t+1} \geq -B \qquad \forall t, \\
&\qquad\quad \lim_{J\to\infty} \left(\frac{1}{1+r}\right)^J a_{t+J+1} = 0 \quad \text{(TVC)}.
\end{aligned}
$$

where B is the borrowing limit. When the borrowing constraint is not binding, we get the usual Euler Equation

$$
\begin{aligned}
& u'(c_t) = \beta(1+r)u'(c_{t+1}) < u'(c_{t+1}) \\
\Rightarrow \quad & c_t > c_{t+1}
\end{aligned}
$$

as long as u is concave. So consumption decreases over time. This implies that savings decrease over time as well, because

$$
\begin{aligned}
& c_{t+1} = \bar{y} + (1+r)a_{t+1} - a_{t+2}, \quad c_t = \bar{y} + (1+r)a_t - a_{t+1} \\
\Rightarrow \quad & 0 > c_{t+1} - c_t = (1+r)(a_{t+1} - a_t) - (a_{t+2} - a_{t+1}) \\
\Rightarrow \quad & a_{t+2} - a_{t+1} > (1+r)(a_{t+1} - a_t) \\
\Rightarrow \quad & \left(\frac{1}{1+r}\right)(a_{t+2} - a_{t+1}) > a_{t+1} - a_t.
\end{aligned}
$$

Then iterating forward,

$$
\begin{aligned}
& 0 = \lim_{J\to\infty} \left(\frac{1}{1+r}\right)^J (a_{t+J+1} - a_{t+J}) > a_{t+1} - a_t \\
& a_t > a_{t+1}.
\end{aligned}
$$

So both consumption and savings decrease; consumption cannot be negative and debt cannot go over the borrowing limit. When the borrowing constraint is binding, then,

$$
a_{t+1} = -B
$$

$$c_t = \bar{y} - rB.$$

So once you hit the borrowing limit, you cannot borrow anymore, continue to pay the interest to your debt, and consume whatever is left over. (If B is the natural borrowing limit $B = \frac{\bar{y}}{r}$, you never hit it - consumption just decreases forever toward zero!)

2.2 $\beta(1+r) = 1$ w/ uncertainty: Prudence and Precautionary Savings

Now $\{y_t\}$ follows a stochastic process with constant mean (i.e., on average the endowment is the same). The Euler equation for the individual is

$$u'(c_t) = \underbrace{\beta(1+r)}_{=1} \underbrace{\mathbb{E}_t u'(c_{t+1}) \gtreqless u'(\mathbb{E}_t c_{t+1})}_{?},$$

If the inequality holds with $>$, then $c_t < E_t c_{t+1}$, so consumption is a submartingale while MU is a supermartingale. By the martingale convergence thm, both c_t and $u'(c_t)$ must converge, the question is where. Turns out that $u'(c_t) \to 0$ (why?), $c_t \to \infty$, and $a_t \to \infty$. (Think about it: if I want to continue to increase consumption, I have to save more every period.) Again, we could have done the same using recursive methods.

When do we have $\mathbb{E}_t u'(c_{t+1}) > u'(\mathbb{E}_t c_{t+1})$? This holds when u' is convex. Hence, u being concave is not enough, or put differently, high risk aversion is not enough even though it may seem like it would. For example, quadratic utility function will not achieve this: assume $u(c) = -c^2$, then $u'(c) = -2c$, so

$$\begin{aligned} -2c_t &= \mathbb{E}_t \left[-2c_{t+1}\right] \\ c_t &= \mathbb{E} c_{t+1}. \end{aligned}$$

and we would get certainty equivalence—i.e., Friedman's PIH in (1) becomes true, not just a conjecture. What we need is u' being convex, or, $u''' > 0$, which we call *prudence*. If the utility function displays prudence, the savings behavior displays *precautionary savings*, so that I save more than I would without uncertainty (certainty equivalence does *not* hold). In fact, this is how Hayne Leland disproved Friedman's PIH.

2.3 $\beta(1+r) < 1$ w/ uncertainty

This is the most important case. Suppose that the endowment process is Markov, i.e., that $y' \sim F(y'|y)$. To make things simpler, suppose that y can only take on a n number of values, so that the Markov process can be expressed as a transition matrix $\Pi_{n\times n}$: each element tells you the probability that the next period endowment is y_j if the current period endowment is y_i. The savings problem is

$$\begin{aligned} v(a, y_i) &= \max_{a' \geq -B} \left\{ u(c) + \beta \mathbb{E}_{y_i} v\left(a', y_j\right) \right\} \quad \text{s.t.} \quad c + a' \leq y_i + (1+r)a \\ &= \max_{a' \geq -B} \left\{ u((1+r)a + y_i - a') + \beta \sum_{j=1}^{n} \pi_{i,j} v(a', y_j) \right\} \end{aligned}$$

where I have just replaced c in the period utility function and explicitly expressed the expectation. Now, let us redefine some variables as follows. Let

$$\begin{aligned} A &= a + B \\ Z &= (1+r)a + B + y_i = (1+r)A + y_i - rB, \end{aligned}$$

because then we can write

$$V(Z, y_i) = \max_{A' \geq 0} \left\{ u(Z - A') + \beta \sum_j \pi_{i,j} V\left((1+r)A' + y_j - rB, y_j\right) \right\}. \tag{2}$$

Note that we still have to write y_i in today's state and y_j in tomorrow's state, because it is needed to know which elements of $\pi_{i,j}$ we should sum over in the expectation. Other than that, it no longer contains any relevant information about how much wealth I have today and how much I can borrow/save; it is all compressed into (A, Z). Here, Z is interpreted as "cash at hand," it is the total amount I can eat taking into consideration how much I can borrow. A is "net investment," it is how much I can set aside for tomorrow.

We can now prove an important result:

THEOREM 1 *Assets are bounded above if absolute risk aversion converges to 0, i.e.*

$$\lim_{c \to \infty} \frac{u''(c)}{u'(c)} = 0.$$

Proof: We can rewrite (2) as

$$\begin{aligned} V(Z) &= \max_{c,A' \in [0,Z]} u(c) + \beta \mathbb{E} V[(1+r)A' + y' - rB] \quad \text{s.t.} \quad c + A' \le Z. \\ &= u(c(Z)) + \beta \mathbb{E} V[(1+r)A'(Z) + y' - rB] \end{aligned}$$

where $c(Z), A'(Z)$ are the optimal allocations for the second maximization problem. Assume u and V are strictly increasing, concave and differentiable. The second maximization problem can be viewed as a standard 2-good utility maximization problem where Z is your wealth, and both c and A' are normal goods. Hence the optimal solutions $c(Z)$ and $A'(Z)$ are both (strictly) increasing in Z.[1]

Now we need

LEMMA 1 *$\exists Z^*$ s.t. for all $Z \ge Z^*$, $Z' \le Z'_{max} = (1+r)A'(Z) + y_{max} - rB \le Z$, where y_{max} is the highest possible realization of income.*

Intuitively, the meaning of the claim is this: suppose my cash-at-hand goes over Z^* today. Then even if I get the highest possible income shock, I will decrease my cash-at-hand tomorrow. Similarly we can define $Z'_{min} = (1+r)A'(Z) + y_{min} - rB$.

Proof: The Euler equation for V is

$$\begin{aligned} u'(c(Z)) &= \beta(1+r)\mathbb{E}u'(c(Z')) \\ &= \beta(1+r)\frac{\mathbb{E}u'(c(Z'))}{u'(c(Z'_{max}))}u'(c(Z'_{max})) \\ &< \frac{\mathbb{E}u'(c(Z'))}{u'(c(Z'_{max}))}u'(c(Z'_{max})). \end{aligned}$$

As $Z \to \infty$, $A'(Z) \to \infty$ since $A'(Z)$ is increasing in Z, so $Z'_{max} \to \infty$. If we can show that $\frac{Eu'(c(Z'))}{u'(c(Z'_{max}))} \to 1$ as $Z'_{max} \to \infty$ we are done, since then

$$\lim_{Z\to\infty} u'(c(Z)) \le \lim_{Z\to\infty} u'(c(Z'_{max}))$$

[1] See Appendix for proof.

so that for some Z^* large enough,

$$
\begin{aligned}
& u'(c(Z)) \leq u'(c(Z'_{max})) \\
\Leftrightarrow \quad & c(Z) \geq c(Z'_{max}) \\
\Leftrightarrow \quad & \quad Z \geq Z'_{max} \quad \geq \quad Z'
\end{aligned}
$$

for all $Z \geq Z^*$, as desired. Now

$$
1 \leq \frac{Eu'(c(Z'))}{u'(c(Z'_{max}))} \leq \frac{u'(c(Z'_{min}))}{u'(c(Z'_{max}))} \leq \frac{u'\big(c(Z'_{max}) - (Z'_{max} - Z'_{min})\big)}{u'(c(Z'_{max}))},
$$

the last inequality since $c(Z'_{max}) - c(Z'_{min}) \leq Z'_{max} - Z'_{min}$ and both c and A' are increasing in Z. Then

$$
\begin{aligned}
1 \leq \frac{u'\big(c(Z'_{max}) - (Z'_{max} - Z'_{min})\big)}{u'(c(Z'_{max}))} &= 1 + \int_0^{Z'_{max} - Z'_{min}} \frac{u''(c(Z'_{max}) - z)}{u'(c(Z'_{max}))} dz \\
&= 1 + \int_0^{Z'_{max} - Z'_{min}} \frac{u'(c(Z'_{max}) - z)}{u'(c(Z'_{max}))} \frac{u''(c(Z'_{max}) - z)}{u'(c(Z'_{max}) - z))} dz \\
&\to 1
\end{aligned}
$$

by the assumption $\lim_{c \to \infty} \frac{u''(c)}{u'(c)} = 0$. *Q.E.D.*

Appendices

Solve

$$\max_{c,a} u(c) + v(a)$$
$$\text{s.t.} \quad c + a \leq W.$$

Now suppose u and v are strictly increasing, concave and differentiable. Attaching the Lagrangian multiplier λ to the constraint, we obtain the f.o.c.'s:

$$u'(c^*) = v'(a^*) = \lambda,$$

so $c^* = f(\lambda)$ and $a^* = g(\lambda)$ where both f and g are decreasing. Plugging this into the budget constraint, we find that $h(\lambda) = W$, where h is decreasing. Hence if W increases, λ decreases, and both c^* and a^* increase.

References

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