Most of these notes are based on Huggett (1993); Aiyagari (1994).

Suppose there are a mass of heterogenous agents who all solve the savings problem

$$\begin{aligned} v(a,y_i) &= \max_{a' \geq -B} \left\{ u(c) + \beta \mathbb{E}_{y_i} V\left(a', y_j\right) \right\} \quad \text{s.t.} \quad c + a' \leq y_i + (1+r)a \\ &= \max_{a' \geq -B} \left\{ u((1+r)a + y_i - a') + \beta \sum_{j=1}^{n} \pi_{i,j} v(a', y_j) \right\} \end{aligned} \tag{1}$$

which we rewrote as

$$V(Z, y_i) = \max_{A' \geq 0} \left\{ u(Z - A') + \beta \sum_j \pi_{i,j} V\left((1+r)A' + y_j - rB, y_j\right) \right\} \tag{2}$$

where

$$\begin{aligned} A &= a + B \\ Z &= (1+r)a + B + y_i = (1+r)A + y_i - rB, \end{aligned}$$

Now suppose we are in an economy with many agents or production (or both), who all solve the same problem. For now, suppose that the aggregate endowment, that is, the total amount that is injected into the economy, is constant. Then we must have in equilibrium (note that in the individual's problem there is no consideration for equilibrium)

1. With no uncertainty, the interest rate must satisfy $\beta(1+r) = 1$
2. With uncertainty, *DARA and prudence*, the interest rate must satisfy $\beta(1+r) < 1$

for there to be a long-run equilibrium with positive but non-infinite consumption and savings. Note that we are making all these statements based on the analysis of a single individual. Of course, if we assume that everyone is the same and think of a representative agent, we are done. But we want to do a bit more.

1. Equilibrium with uncertainty but constant aggregate endowments (Huggett, 1993)

Now suppose there are a continuum of individuals of mass 1; i.e., the entire population can be described as a probability distribution. Each individual solves (1). But now instead of assuming a given interest rate, we assume it is determined in a market where individuals in which individuals can trade bonds. There are no foreigners nor government who can supply/demand additional bonds; all bonds are supplied and demanded by the individuals within the economy.

Given the policy function for savings, consumption is just the residual, which is also a function of the state $c(a, \epsilon)$. A *stationary recursive competitive equilibrium* is defined as

Definition 1 *A stationary recursive competitive equilibrium is a distribution over assets and endowments $F(a, y_i)$, policy functions $c, a' : [-B, \bar{a}] \times \{y_1, \cdots, y_n\} \mapsto [-B, \bar{a}]$, and an interest rate r s.t.*

1. *The policy functions $c(a, y), a'(a, y)$ solves the individual problem (1),*
2. *Goods and bond markets clear:*

$$\int a dF(a, y_i) = 0,$$

3. *The equilibrium is stationary, i.e. the distribution F satisfies $\int \Gamma dF = F$, or*

$$\sum_{i=1}^{n} \pi_{i,j} \int_{a'(\tilde{a}, y_i) \leq a} dF(\tilde{a}, y_i) = F(a, y_j).$$

where Γ is the equilibrium transition function.

where I have ignored the goods market by Walras' law. Now let us think about what stationarity means. For the endowments it's standard, since we assumed it is a Markov chain; we just need to find a probability vector $p_{1\times n}$ s.t.

$$p\Pi = p \quad \Rightarrow \quad p \cdot (I - \Pi) = 0$$

i.e., p is the eigenvector of Π with associated eigenvalue 1. Note that this also means that if the economy begins at p, aggregate endowments, i.e., the total sum of endowments received by all individuals in this economy, remains constant. Moreover, if Π is ergodic (and aperiodic), the stationary distribution can be found by starting from *any* initial distribution and iterating forward, i.e. if we start from p_0,

$$\begin{aligned} & p_1 = p_0\Pi, \qquad p_2 = p_1\Pi = p_0\Pi^2, \qquad \ldots \qquad p_t = p_{t-1}\Pi = \ldots = p_0\Pi^{t-1} \\ \Rightarrow \qquad & \lim_{t\to\infty} p_t = \lim_{t\to\infty} p_0\Pi^t = p. \end{aligned}$$

Things are a bit trickier once we also consider the distribution over savings. Note that there are two sources of heterogeneity: one is exogenous, given by y_i and its associated transition matrix. The other is endogenous, the next period savings choices individuals make depending on their current savings and endowment. It is *NEVER* the case that there will just be a N levels of savings if there are N levels of y_i: how much savings an individual has will depend on the entire history of shocks, not just the current one.

So technically, the next period distribution over $s = (Z', y_j) \in S$ can be anything depending on this period's distribution over (a, y_i), which we will call F_t. *If* the transition function Γ over F_t is also Markov, then, we know that there exists a stationary distribution over S as well, which we are calling F. Moreover, starting from any distribution F_0, we will have that

$$\lim_{t\to\infty} F_t = \lim_{t\to\infty} \int \Gamma^t dF_0 = F. \tag{3}$$

So stationarity means that the economy has run for long enough that although individuals are switching their savings decisions and getting hit by different shocks, the aggregate distribution looks the same. This is the corresponding notion to a steady state in a representative agent model: a stationary distribution with heterogeneous agents.

But how do we know such a distribution exists? First, remember that if $\beta(1+r) < 1$, then we know that we can always find a $\overline{Z}$ s.t. $Z \in (0, \overline{Z})$ in (2) for any (Z, y_i) s.t. $Z < \overline{Z}$. Moreover since $y_i \in Y$ is assumed to be a countably finite set, the state space S is compact. Now, define the Borel σ-algebra of S, Ω: don't worry, this is just the set of all appropriate subsets of S. Then we can formally define the transition function in the equilibrium definition 3:

$$\Gamma : S \times \Omega \to [0,1] \quad \text{s.t.} \quad \Gamma(s, \omega) = \sum_{y'} \pi(y'|y) \cdot \mathbb{1}\left\{(a'(a,y), y') \in \omega\right\}.$$

and formally restate the stationarity condition as

$$\int_{s\in S}\Gamma(s,\omega)dF(s)=F(\omega)\quad\forall\omega\in\Omega.$$

THEOREM 1: MONOTONE MIXING CONDITION *Suppose there exists $\hat{s}\in S$ for which we can find $T\geq 1$ and $\epsilon>0$ s.t.*

$$\Gamma^T(s_M,\{s':s'\leq\hat{s}\})>\epsilon\quad\textit{and}\quad\Gamma^T(s_m,\{s':s'\geq\hat{s}\})>\epsilon,$$

where (s_m,s_M) is the min and max in S. Then Γ is ergodic (and aperiodic). Therefore, an stationary distribution exists; moreover any distribution converges to the stationary distribution.

So by starting from any distribution F_0 over (a,y_i), the stationary distribution is found by iterating into the indefinite future. This is just to say that the stationary equilibrium is well defined; how we actually find it numerically will come in later classes.

In our application, $s_m=(y_1-rB,y_1)$ and $s_M=(\overline{Z},y_N)$. These are the "worst" and "best" states an individual can be in. What MMC means is the following. Suppose we start off two individuals, one with s_m and the other s_M. If we can find a cutoff, $\hat{s}=(\hat{Z},\hat{y})$, s.t. in the *same* finite time, both the probability of the worst-off guy being in a better state than $\hat{s}$ *and* the best-off guy being in a worse state than $\hat{s}$ are positive, we can find a stationary distribution. But if you think about it, this is trivially true: just keep hitting the s_m guy with y_N-shocks and s_M guy with y_1-shocks, and in finite time the poor guy will approach $\overline{Z}$ and the rich guy will hit the borrowing constraint—from the monotonicity of the savings function.

Note that stationarity induced from such a transition function (i.e., ergodic and aperiodic) also means that any given individual will pass through all the possible states, and the cumulative distribution of which states the *individual* falls in *over time* is equal to the cross-sectional stationary distribution: This is implied by (3).

Now let us think about how the equilibrium interest rate would look like under the stationary distribution. First, we know that bond markets clear, and since F is stationary, the market clearing condition 2 is the same in every period. In other words, the sum of savings is always the same—even though individual savings are changing over time. Hence, the interest rate does not change over time (just like in a steady state), i.e. $r_t\equiv r$. Second, we know that **as long as the period utility exhibits prudence**, if $\beta(1+r)=1$, assets will diverge to infinity on average, and by assumption, no one's assets can go below $-B$. If B is the natural borrowing constraint, no matter what the interest rate is, it will never be hit. So we can draw an aggregate savings function with r on the y-axis and $A = \int a dF(a, y_i)$ on the x-axis, and the equilibrium interest rate is simply the y-intercept. That is, we find the interest rate as the intersect of asset demand $A(r)$ and supply, 0.

What happens to $A(r)$ as we vary B, the borrowing constraint? We can safely constraint the interest rate between $(-1, 1/\beta - 1)$, since at the lower bound no one will save, and at the upper bound aggregate savings diverge to infinity. The Euler equation was

$$u'(c) \geq \beta(1+r) \sum_j \pi_{i,j} u'(c'), \quad \text{with equality if} \quad A' \geq 0.$$

Since we defined $c = Z - A'$ and $Z = (1+r)A + y_i - rB$,[1]

1. whenever the constraint is not binding we have that $A'(Z)$ will be the same as long as Z is the same, no matter what the level of B. So it seems if we look at the aggregate savings function $A(r; B)$, it seems it should just be horizontally shifted by B.

2. But this is NOT the case, since Z itself depends on B. Another way to see this is that even if today's state is the same, tomorrow's state, $Z' = (1+r)A' + y_j - rB$ in (2), depends on B. So the optimal saving rule $A'(Z; B)$ implicitly depends on B.

3. However, when $r = 0$, it does not: so only in this case, $a'(a + B; 0) = a'(a; B) + B$.

4. Moreover, when $B > y_1/R$, i.e., is tighter than the natural borrowing constraint, everyone hits the constraint in finite time, and there will always be a positive mass of constrained individuals in the stationary distribution no matter the value of r.

5. Besides the natural borrowing constraint, the $B = 0$ case is also of special interest. In this case, no one is allowed to borrow, so in equilibrium, no one will be saving either. In order to prevent people from saving, the Euler equation must always hold with inequality at the equilibrium interest rate:

$$r \leq \min \left\{ \frac{u'(c)}{\beta \sum_j \pi_{i,j} u'(c')} \right\} - 1.$$

[1] Sorry about the abuse of notation—I am using A for both the "net savings" in the individual's problem, and for aggregate savings...

2. Aggregate savings and production (Aiyagari, 1994)

Now suppose we also assume there is a representative firm that rents the individuals' savings, which they use as capital (we can think of savings as capital by no arbitrage), and hire them at a wage of w. However, individuals are now shocked not in terms of their endowments, but in terms of their labor productivity ϵ.

Specifically, suppose that ϵ can only take on n values, follows a Markov chain described by Π, and further assume that the stationary distribution of Π, which I will again call p, satisfies $\mathbb{E}\epsilon = \sum_{i=1}^{n} p_i \epsilon_i = 1$. The individual's problem is

$$
\begin{aligned}
v(a, \epsilon_i) &= \max_{a' \geq -B} \left\{ u(c) + \beta \mathbb{E}_{\epsilon_i} v(a', \epsilon_j) \right\} \quad \text{s.t.} \quad c + a' = w\epsilon_i + (1+r)a \\
&= \max_{a' \geq -B} \left\{ u((1+r)a + w\epsilon_i - a') + \beta \sum_{j=1}^{n} \pi_{i,j} v(a', \epsilon_j) \right\}.
\end{aligned} \tag{4}
$$

The only difference from (1) is that I wrote $w\epsilon_i$ instead of y_i.

The representative firm's problem is same as always, it solves

$$
\max_{K,L} F(K, L) - RK - wL,
$$

so $F_1 = R$ and $F_2 = w$. If we assume capital depreciates at a rate of δ *during* production, $R = r + \delta$ in competitive equilibrium. There are several ways we can explain this:

1. Suppose that competitive banks collect a from the consumers promising an interest of r. It then lends this out to firms as capital at a rate of R. While firms use the capital it depreciates by a rate of R. So, bank profits are $R - \delta - r$ per unit of deposit collected from the consumers. Since banks are competitive, if any of the banks make any profit by this intermediation, another bank can get more profit by offering a slightly smaller R to the firm and/or a slightly larger r to the consumers; hence $R = r + \delta$ in a competitive equilibrium.

2. Suppose we assumed that consumers can in fact hold both bonds and capital. If either has a higher return, there will either hold only bonds or only capital. However, net bond supply must be zero since no one else is supplying them, while if capital is zero there would be no production at all. In order for the consumers to hold on to capital and zero bonds (no arbitrage), $R = r + \delta$.

In any case, the reason that $R = r + \delta$ is just the same as in the Ramsey growth model. We can now define a recursive equilibrium with production:

DEFINITION 2 *A stationary recursive competitive equilibrium with production is a distribution over assets and labor productivity* $F(a, \epsilon_i)$*, a policy function* $c, a' : [-B, \bar{a}] \times \{\epsilon_1, \cdots, \epsilon_n\} \mapsto [-B, \bar{a}]$*, aggregate capital and labor allocations* (K, L)*, and a collection of prices* (R, r, w) *s.t.*

1. *The policy functions* $c(a, \epsilon), a'(a, \epsilon)$ *solves the individual problem (4),*
2. *The representative firm maximizes profits,*
3. *Goods, asset and labor markets clear:*

$$\int a dF(a, \epsilon_i) = K, \qquad \int \epsilon_i dF(a, \epsilon_i) = L = 1,$$

   and $R = r + \delta$*,*

4. *The equilibrium is stationary, i.e. the transition function satisfies* $\int \Gamma dF = F$*, or*

$$\sum_{i=1}^{n} \pi_{i,j} \int_{a'(\tilde{a}, \epsilon_i) \leq a} dF(\tilde{a}, \epsilon_i) = F(a, \epsilon_j).$$

where I have again ignored the goods market. Again, nothing is changing over time in aggregate, so from the point of view of the representative firm, K and L are remaining constant—hence in a stationary equilibrium, (R, r, w) are also constant. Furthermore, since we assumed that labor is inelastically supplied and that the mean of ϵ is 1, there is not anything interesting going on with labor. Once we know the steady state value of $k = K/L$ and interest rate r, and the wage rate w would just depend on these two objects:

$$w = F_2(K, L) = f(k) - kf'(k) = f(k) - k(r + \delta).$$

EXERCISE 1 *Show that if* $F(K, L) = K^\alpha L^{1-\alpha}$*, and the representative firm maximizes profits,*

$$w = (1 - \alpha) \cdot \left(\frac{R}{\alpha}\right)^{\frac{\alpha}{1-\alpha}}$$

in a stationary equilibrium. So with specific functional forms for F*, we don't even need to know the steady state value of* k *to get* w*—we only need to know* r*.*

So as in the previous section, we can draw an aggregate savings function $A = \int a dF(a, \epsilon_i)$ as a function of the interest rate. But now, the equilibrium is not when this function crosses 0, which was required for bonds to be in zero net supply, but when it crosses $r_d = f'(k) - \delta$, the firm's demand schedule as a function of the interest rate.

Note that the interest rate is lower higher than in the endowment economy, because the existence of the firm creates a demand for consumers to supply more capital. Also note that in both cases, the interest rate must always be lower than $1/\beta - 1$. We will soon learn that this is not just because of uncertainty, but because of the lack of insurance markets—hence the term "incomplete markets."

c.f. If interested, read the chapter "Incomplete Market Models" in Ljungqvist and Sargent (2004) for many more applications, including inside and outside money.

References

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