Many of you may alread know this, so I won't spend too much time here. However, remember the implications for asset pricing, debt-constrained markets, and be careful to remember the differences of an Arrow-Debreu vs. Recursive equilibrium, and Arrow-Debreu vs. Arrow securities, especially for OLG economies (which comes in Cezar's course).

Setup Consider the following endowment economy in Ljungqvist and Sargent (2004) Chapter 8. Extending all these results to a production economy is not trivial but a direct extension, in Chapter 12.

1. Discrete time $t = 1, 2, \cdots, \infty$.
2. Stochastic event $s_t \in S \Rightarrow$ histories $s^t = [s_0, s_1, \cdots, s_t]$.
3. Probabilities $\pi_t(s_t)$ over histories, conditional probabilities $\pi_t(s^t|s^\tau)$, $\pi_0(s_0) = 1$ given.[1]
4. Preferences for each individual i given contingent consumption plans $c = \{c_t^i(s^t)\}_{t=0}^{\infty}$:

$$U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u\left(c_t^i(s^t)\right) \pi_t(s^t).$$

   with Inada conditions on the utility function.[2]

5. Stochastic individual endowments $y_t^i(s^t)$; resource constraint with no aggregate uncertainty:

$$\sum_i c_t^i(s^t) \le \sum_i y_t^i(s^t) = Y_t,$$

   for all t, s^t. So the sum of endowments Y_t may vary over time, but only non-stochastically.

Henceforth we drop subscripts i unless imperative.

EXERCISE 1 *(Ljungqvist and Sargent (2004), pp.212-213). Assume there is a planner that attaches λ_i weights to each individual, with $\sum_i \lambda_i = 1$. Show that the efficient allocation depends only on (λ_i, Y_t) and the utility function, but nothing else.*

Arrow-Debreu Securities The idea of an Arrow-Debreu equilibrium is simple and elegant. Recall the standard general equilibrium results you learned in micro; now, instead of having horizontally differentiated goods, you just have history-contingent goods. At time 0, you buy and sell goods in each history-contingent market, just like in a standard general equilibrium. So all the results, in particular the 1st and 2nd Welfare Theorems, carry over.

The only difference is that **instead of actually trading the goods, you exchange claims to the goods**. All trade happens **only at time** 0 and everyone just uses their claims from now to eternity. Denote the price of these claims by $q_t^0(s^t)$. This is confusing:

[1] As explained in the book, this assumption is not necessary, but let's just assume it.
[2] As explained in the book, we could do with a less restrictive assumption, but let's just assume it.

- Subscript t is for when the consumption occurs.
- s^t is the history of realizations that must happen for this particular claim to be exercised.
- Superscript 0 is for when the trade occurs. I just said all trade occurs at time 0, so why do we need this? Later, for sequential trading.

Households maximize utility subject to individual budget constraints

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)c_t(s^t) \leq \sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)y_t(s^t),$$

leading to

$$\beta^t u'\left(c_t^i(s^t)\right)\pi_t(s^t) = \mu_i q_t^0(s^t)$$

where μ_i's are Lagrangean multipliers attached to the budget constraint. Along with the market clearing condition

$$\sum_i c_t^i(s^t) \leq \sum_i y_t^i(s^t) = Y_t$$

we have an equilibrium. So if we normalize $q_0^0(s_0) = 1$, equilibrium prices are

$$q_t^0(s^t) = \frac{\beta^t u'\left(c_t^i(s^t)\right)\pi_t(s^t)}{u'\left(c_0^i(s_0)\right)\pi_0(s_0)}.$$

This simply implies:

i. Price=MRS due to individual utility maximization subject to budget constraint.

ii. MU ratio equalization for each good across all agents due to market clearing: in the Arrow-Debreu world, this means MU ratios are equalized across all histories.

The idea is simple, the consequences and applications are endless. For example:

1. You can price any asset. Suppose there is a tree that drops $\{d_t(s^t)\}_{t=0}^{\infty}$ in every period, but still there is no aggregate uncertainty (the total endowment in the economy is known). The price of this tree at time 0 is

$$p_0^0(s_0) = \sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)d_t(s^t).$$

EXERCISE 2 *LS 8.7.3: What is the price of a t-period riskless bond?*

A riskless bond promises a face value at time t, for *any* possible realized history. If that face value is F,

$$P_{bond}^0 = F \cdot \sum_{s^t} q_t^0(s^t),$$

so $\sum_{s^t} q_t^0(s^t)$ is the inverse of the t period risk-free interest rate.

2. You can model limited commitment (Kehoe and Levine (1993)). In the Arrow-Debreu world, if you asked any of the guys if they are happy with their claims at any given date $\tau > 0$, they will say yes—i.e., they wouldn't trade even if a claims market opens again at time τ. However, if you ask them if they want to *stay* in the market, there will be guys who say no. To prevent this, you can assume a participation constraint in the consumer's utility maximization problem

$$\sum_{t=\tau}^{\infty}\sum_{s^t}\beta^t u\left(c_t(s^t)\right)\pi_t(s^t|s^\tau) \geq \sum_{t=\tau}^{\infty}\sum_{s^t}\beta^t u\left(y_t(s^t)\right)\pi_t(s^t|s^\tau)$$

for all individuals, for *all* $\tau \geq 0, s^\tau$. This doesn't mean that trading takes place after time 0—there are still no claims actually being traded at the prices $q_t^\tau(s^t)$ in equilibrium, these are merely implied. However, in the KL-equilibrium, everyone is happy to stay in the market. Notice that the constraint is convex in history-contingent consumption, so the consumer's problem is well defined.

EXERCISE 3 *LS 8.7.4: Prove that no trade occurs in subsequently opened markets in an Arrow-Debreu equilibrium.*

EXERCISE 4 *Prove that the participation constraint in the KL economy leaves the consumption possibility set convex. (see paper for proof)*

Arrow Securities Once you understand the AD-equilibrium, it's easy to generalize it to the more complicated but interesting case of sequential trading markets—i.e., a market opens every period, and people trade claims only on next period consumption. We just need one more assumption on a debt limit, i.e., how much of these claims an agent can buy. Define the natural borrowing limit

$$A_t^i(s^t) = \sum_{\tau=t}^{\infty}\sum_{s^\tau|s^t} q_\tau^t(s^\tau)y_\tau^i(s^\tau)$$

for all t following history s^t. Now, think of a guy making a decision at time t following a particular history s^t. There is a claims market *every period, but only for next period consumption*. Then

1. this guy wants to insure himself by buying some assets for each possible realization $s_{t+1}|s^t$. So he wants to trade as many claims as possibly realizable states, so he faces just as many debt limits.

2. For each state-contingent claim, his (currently unrealized) constraint for selling claims is $A_{t+1}^i(s_{t+1}|s^t)$. To satisfy his optimal consumption plan, he sells exactly

$$a_{t+1}^i(s_{t+1}|s^t) \geq -A_{t+1}^i(s_{t+1}|s^t) + \sum_{\tau=t+1}^{\infty}\sum_{s^\tau|s^{t+1}} q_\tau^{t+1}(s^\tau)c_\tau^i(s^\tau).$$

claims, for all possible realizations of s_{t+1} given s^t.

3. The one period ahead prices of these claims are intuitively (but not so obviously)

$$Q_t(s_{t+1}|s^t) = \frac{\beta u'\left(c_{t+1}(s^{t+1})\right)\pi_t(s_{t+1}|s^t)}{u'\left(c_t(s^t)\right)} = \frac{q^0_{t+1}(s^{t+1})}{q^0_t(s^t)}.$$

This is all nice and all, but terribly abstract and complicated. So we impose a Markov structure, see separate notes for a brief review. This admits a nice recursive structure:

$$V(a,s) = \max_{c,a'(s')}\left\{u(c) + \beta\sum_{s'} V(a'(s'),s')\pi(s'|s)\right\}$$

subject to

$$c + \sum_{s'} Q(s'|s)a'(s') \leq y(s) + a$$
$$a'(s') \geq -A(s'),$$

where $A(s)$ is also recursive (i.e., the function $A(s)$ depends only on s and not time):

$$A(s) = y(s) + \sum_{s'} Q(s'|s)A(s').$$

There are as many markets to clear as possibly realizable states s':

$$\sum_i a'_i(s') = 0.$$

for all s'.

Although easier on the eyes, the recursive equilibrium is a much more fragile object in terms of existence, etc., than the AD equilibrium, but we won't deal with this in this course. Now, even more than the AD equilibrium, the applications are endless:

EXERCISE 5 *What's the price of a 1-period riskless bond? Of a stock with a random (recursively representable) dividend $\{d(s'|s)\}$? For a 1-period forward contract that requires the holder to buy one 1-period bond with strike price K? A 1-period forward contract that requires the holder to buy one share of stock with strike price K? What about j periods?*

and more in homework.

References

KEHOE, T. J., AND D. K. LEVINE (1993): "Debt-Constrained Asset Markets," *Review of Economic Studies*, 60(4), 865–88.

LJUNGQVIST, L., AND T. J. SARGENT (2004): *Recursive Macroeconomic Theory, 2nd Edition*, vol. 1 of *MIT Press Books*. The MIT Press.