SANG YOON (TIM) LEE

The setup I use in these notes will be the following—mostly taken from Resnick (1992); Ljungqvist and Sargent (2004).

- state space $S = \{s_1, s_2, \cdots\}$
- stochastic process $\{x_t\}$, $t = 0, 1, \cdots$
- initial distribution: the row vector $\mu = (\mu_1, \mu_2, \cdots)$, $\mu_k \geq 0$, $\sum_{k=0}^{\infty} \mu_k = 1$ s.t.

$$\mu_k = P[x_0 = s_k]$$

- transition matrix $P = (p_{ij}, i \geq 0, j \geq 0)$ s.t.

$$p_{ij} = P[x_{t+1} = s_j | x_t = s_i]$$

DEFINITION 1 *$\{x_t\}_{t\geq 0}$ is said to have the Markov property if for all $k = 1, \cdots, t$*

$$P[x_{t+1}|x_t, x_{t-1}, \cdots, x_{t-k}] = P[x_{t+1}|x_t].$$

Such a process is called a Markov chain.

Easy to verify that for a Markov chain with initial distribution μ and transition matrix P,

$$P[x_0 = i_0, x_1 = i_1, \cdots, x_k = i_k] = \mu_{i_0} p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k}.$$

Conversely, if (1) holds for all k and $(i_0, \cdots, i_k)$ for some transition matrix P and initial distribution μ, the Markov property also holds.

P describes the one-step transition probability from state to state. Once we have P, it is easy to obtain the n-step transition probability:

$$p_{ij}^{(n)} = P[x_n = j | x_0 = i].$$

With these basics in mind, we move on to some important properties of Markov chains:

DEFINITION 2 *For $i, j \in S$, we say j is accessible from i, written $i \to j$, if $\exists n \geq 0$ s.t. $p_{ij}^{(n)} \geq 0$.*

DEFINITION 3 *States i and j communicate if $i \leftrightarrow j$.*

The state space S may now be decomposed into disjoint exhaustive equivalence classes. Pick a state, say s_0, and put all states communicating with s_0 in a class, say C_0. Then pick a state in $S \setminus C_0$, call it s_i, and put it and all states communicating with i into another class which we name C_1. Continuing in this manner we have

$$C_i \bigcap C_j = \phi, \quad \text{and} \quad \bigcup_i C_i = S.$$

DEFINITION 4 *A Markov chain is irreducible if the state space consists of only one class; i.e., for any* $i, j \in S$ *we have* $i \leftrightarrow j$.

DEFINITION 5 *State i is recurrent if the chain returns to i with probability 1 in a finite number of steps. Otherwise the state is transient. To be more precise, let* τ_i *be the hitting time of state i, and* $P_i[\cdot] = P[\cdot | x_0 = i]$. *Then state i is recurrent if*

$$P_i[\tau_i < \infty] = 1.$$

State i is positive recurrent if

$$E_i(\tau_i) < \infty.$$

Otherwise, it is null recurrent.

DEFINITION 6 *The period of state i is*

$$d(i) \equiv \gcd\{n \geq 1 : p_{ii}^{(n)} > 0\}.$$

If $d(i) = 1$, *we say that i is aperiodic.*

Now let the row vector π_t be the unconditional probability distribution over S at time t. We have

$$\pi'_{t+1} = \pi'_t P.$$

DEFINITION 7 $\pi = (\pi_j, j \geq 0)$ *is called a stationary distribution or invariant distribution for the Markov chain with transition matrix P if*

$$\pi' = \pi' P,$$

i.e., $\pi_j = \sum_{k \in S} \pi_k p_{kj}, j \in S$.

So if a Markov chain starts at the invariant distribution π, it is stationary: for any integers $m \geq 0$ and $k \geq 0$ have

$$P[x_0, \cdots, x_m] = P[x_k, \cdots, x_{k+m}],$$

that is, the two vectors have the same joint distributions whatever the length m of the vector and whatever the translation k may be.

We want to know about the uniqueness and existence of invariant distributions: there exists a unique invariant distribution if the Markov chain is positive recurrent and irreducible; put simply, starting from any state s_i, there must be positive probability that *all* states s_j are reached within finite time. This property is also called *ergodicity*: an irreducible, recurrent Markov chain with an invariant distribution is called ergodic. Ergodicity is required for the *SLLN for Markov chains*:

Theorem 1 *Suppose $\{x_t\}$ is ergodic. Let $f : S \to \mathbf{R}$ be a function that satisfies*

$$\sum_{k \geq 0} |f(s_k)| \cdot \pi_k < \infty.$$

Then, regardless of the initial distribution,

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} f(x_t) \to \sum_{k \geq 0} f(s_k) \cdot \pi_k$$

with probability 1 (almost surely).

In many economic applications, we are interested in the existence of a steady state distribution:

Theorem 2 *Suppose the Markov chain is ergodic and aperiodic. Then*

$$\lim_{n \to \infty} p_{ij}^{(n)} = \pi_j, \quad \forall i, j \in S.$$

References

Ljungqvist, L., and T. J. Sargent (2004): *Recursive Macroeconomic Theory*. MIT Press, Cambridge, MA, 2nd edn.

Resnick, S. I. (1992): *Adventures in Stochastic Processes*. Birkhäuser, Boston.