The subscripts on continuous time variables mean that they are functions of time. Derivatives without arguments are evaluated at the optimal values. Most of these notes are just summaries or applications of existing references (must be added).

1. Savings Problem in Continuous Time

Later, we will define markets where consumers have to go out and trade with the (representative) firm(s) and others, and solve for a **competitive equilibrium**. First we just solve the individual's problem. To build intuition, we start from a finite T (life-cycle model).

$$\begin{aligned} W(s,a) = &\max_{\{c_t\}} \int_s^T e^{-\rho(t-s)} u(c_t) dt \\ &\text{s.t.} \quad c_t + \dot{a}_t = w_t + r_t a_t, \\ &\qquad a_s = a \text{ given}, \quad a_T \geq 0. \end{aligned}$$

where $u(c)$ satisfies standard regularity conditions (does not have to CRRA for now). To describe the optimal solution—the evolution of (c_t, a_t) as *functions of time*—we will be working with the (current value) **Hamiltonian**:

$$\mathcal{H}(t, c_t, a_t, \lambda_t) = u(c_t) + \lambda_t (w_t + r_t a_t - c_t)$$

Sufficient conditions for optimality (which are similar to f.o.c.'s for a Lagrangian) are:

$$c_t: \quad \frac{\partial \mathcal{H}}{\partial c_t} = 0 \qquad \Rightarrow \qquad u'(c_t) = \lambda_t, \tag{1a}$$

$$a_t: \quad \frac{\partial \mathcal{H}}{\partial a_t} = -\dot{\lambda}_t + \rho\lambda_t \qquad \Rightarrow \qquad -\dot{\lambda}_t = \lambda_t (r_t - \rho), \tag{1b}$$

$$\lambda_t: \quad \frac{\partial \mathcal{H}}{\partial \lambda_t} = \dot{a}_t \qquad \Rightarrow \qquad \dot{a}_t = w_t + r_t a_t - c_t, \tag{1c}$$

along with boundary conditions that depend on our assumptions on (T, a_T). Here, c_t is the control, a_t is the state, and λ_t the costate. The costate basically plays the role of a Lagrangian multiplier, which is why we denote it by λ_t. To understand the Hamiltonian, and set $T = \infty$, we need to understand a bit of optimal control.

Heuristic results Before that, let's look at some heuristic derivations. The discrete-time version of our problem is

$$\max_{\{c_t, a_{t+1}\}_{t=s}^{T}} \sum_{t=s}^{T} \beta^{t-s} u(c_t)$$
$$\text{s.t.} \quad a_{t+1} - a_t = w_t + r_t a_t - c_t, \quad t = s, \cdots, T,$$
$$a_s > 0 \text{ given}, \quad a_{T+1} \geq 0.$$

Set the current value Lagrangean with multipliers $\beta^t \lambda_t$ for each $t = s, \cdots, T$:

$$\mathcal{L}\left(\{c_t, a_{t+1}, \lambda_t\}\right) = \sum_{t=s}^{T} \beta^{t-s} \left\{u(c_t) + \lambda_t \left[w_t + r_t a_t - c_t - (a_{t+1} - a_t)\right]\right\}$$

F.o.n.c.'s:

$$c_t : \quad u'(c_t) = \lambda_t$$
$$a_{t+1} : \quad -(\lambda_{t+1} - \lambda_t) = \lambda_t[1 - /\beta(1 + r_t)]$$
$$\lambda_t : \quad a_{t+1} - a_t = w_t + r_t a_t - c_t$$

Now let's instead consider dt increments of time rather than "1." Note that a_t is a stock variable while all other's are flows. Clearly λ_t is also a stock variable, since it is chosen dynamically. Hence these become the "state" and "co-state" variables. Letting $\rho = (1 - \beta)/dt$ the discount rate, we get

$$\mathcal{L}\left(\{c_t, a_{t+dt}, \lambda_t\}\right) = \sum_{t=s}^{T} (1 - \rho dt)^{t-s} \left\{u(c_t)dt + \lambda_t \left[w_t dt + r_t dt \cdot a_t - c_t dt - (a_{t+dt} - a_t)\right]\right\}$$

F.o.n.c.'s:

$$\begin{aligned} u'(c_t) &= \lambda_t \\ -(\lambda_{t+dt} - \lambda_t) &= \lambda_t[1 - 1/(1-\rho dt)(1+r_t dt)] = \lambda_t \cdot \frac{(r_t - \rho)dt - \rho r_t dt^2}{(1-\rho dt)(1+r_t dt)} \\ a_{t+dt} - a_t &= w_t dt + r_t dt \cdot a_t - c_t dt \end{aligned}$$

and then dividing both sides and taking the limit as $dt \to 0$ gives you the optimality conditions. Conversely, suppose there exists a function $V(s,a)$ with which we can "discretize" the continuous time problem recursively:

$$V(s,a) \geq \int_0^{ds} e^{-\rho\tau} u(c_{s+\tau}) d\tau + e^{-\rho ds} \cdot V(s+ds, a + (w_s + r_s a - c_s)ds$$

and taking a 1st order Taylor expansion around $ds = 0$, dividing by ds and sending $ds \to 0$ (we'll be doing a lot of this later), we obtain

$$0 \geq u(c_s) - \rho V(s,a) + V_s(s,a) + V_a(s,a) \cdot (w_s + r_s a_s - c_s)$$

with equality at the optimum. If V is well-behaved, the c_s that maximizes the RHS satisfies $u'(c_s) = V_a$. This equation at the optimum is the "Hamilton-Jacobi-Bellman" equation. Using the Hamiltonian, the HJB can be written as

$$\rho V(t,a) = V_t(t,a) + \mathcal{H}^*(t, a, V_a(t,a))$$

where

$$\mathcal{H}^*(t, a, V_a(t,a)) = \max_c \mathcal{H}(t, a, c, V_a(t,a)),$$

so we know that c_t satisfies $u'(c) = V_a$. The common economic interpretation is as follows. The LHS is the flow payoff whose present discounted value equals V if discounted at rate ρ; it is a "dividend" you receive from having a_t at time t. The RHS is the instantaneous return of holding on to a_t at time t. The HJB equation states that these two objects should be the same; in economic terms, it is a no-arbitrage condition.

2. Deterministic Control

Consider the slightly more general control problem

$$W(s,a) = \max_{\{c_t\}} J(s,a,\{c_t\}) = \max_{c_t} \int_s^T U(t,a_t,c_t)dt + Z(T,a_T)$$
$$\text{s.t.} \quad \dot{a}_t = f(t,a_t,c_t), \quad a_s = a \text{ given}, \quad (T,a_T) \in C \subset \mathbb{R} \times \mathbb{R}^N,$$

where $a : \mathbb{R} \mapsto \mathbb{R}^N$ is the $N \geq 1$ dimensional state, and $c : R \mapsto \mathbb{R}^M$ is the $M \geq 1$ dimensional control.

This is not as easy as it looks because of issues with differentiability and so forth. We will ignore all that and simply assume that all objects are always well-defined. For more rigorous treatments, refer to optimal control text books such as . What we want to obtain is the optimal control function as a function of time, c_t, from time s to T.

The problem states that at time T we get a terminal value ("scrap value") Z, but we are constrained to land on the target set C. For example, if C is a singleton, we have a fixed boundary problem.

2.1 Fixed Terminal Time/State

Let us first assume that T and a_T are not choice variables, but given as exogenous boundary conditions. In this case, we only need to make sure that a_t lands in C at time T. For *any, arbitrary* piecewise differentiable function of t, $\lambda : \mathbb{R} \mapsto \mathbb{R}^N$, we can write the objective function as

$$J(s,a;\{c_t\}) = \int_s^T \{U(t,a_t,c_t) + \lambda_t[f(t,a_t,c_t) - \dot{a}_t]\}\, dt + Z(T,a_T)$$

Integrate $\int_s^T \lambda_t \dot{a}_t dt$ by parts:

$$\int_s^T \lambda_t \dot{a}_t dt = \lambda_T a_T - \lambda_s a_s - \int_s^T \dot{\lambda}_t a_t dt$$

which is well defined by the piecewise differentiable assumption on λ_t. We then have

$$J = \int_s^T \{U(t, a_t, c_t) + \lambda_t[f(t, a_t, c_t) + \dot{\lambda}_t a_t]\} ds + Z(T, a_T) + \lambda_s a_s - \lambda_T a_T$$

We want to characterize necessary conditions for the optimum. So suppose (a_t^*, c_t^*) attain the optimum, and that (a_t, c_t) is any other feasible path. Let

$$\Delta_a(t) = a_t - a_t^*, \qquad \Delta_c(t) = c_t - c_t^*, \qquad \Delta_J(s, a; \{c_t\}) = J(s, a, \{c_t\}) - J^*.$$

It follows that

$$\Delta_J = \int_s^T \left\{U(t, a_t, c_t) - U(t, a_t^*, c_t^*) + \lambda_t[f(t, a_t, c_t) - f(t, a_t^*, c_t^*)] + \dot{\lambda}_t \Delta_a(t)\right\} dt$$

where the last terms of disappear as $a_t \to a_t^*$. A 1st order Taylor expansion around (a_t^*, c_t^*) gives

$$\begin{aligned}\Delta_J \approx &\int_s^T [U_a(t, a_t^*, c_t^*) + \lambda_t f_a(t, a_t^*, c_t^*) + \dot{\lambda}_t]\Delta_a(t) dt \\ &+ \int_s^T [U_c(t, a_t^*, c_t^*) + \lambda_t f_c(t, a_t^*, c_t^*)]\Delta_c(t) dt\end{aligned}$$

At a maximum, $\Delta_J = 0$. So pick λ_t^* (as long as it's piecewise diff.), as well as the control c_t^* to enforce this, i.e.

$$\begin{aligned}-\dot{\lambda}_t^* &= U_a(t, a_t^*, c_t^*) + \lambda_t^* f_a(t, a_t^*, c_t^*) \\ 0 &= U_c(t, a_t^*, c_t^*) + \lambda_t^* f_c(t, a_t^*, c_t^*).\end{aligned}$$

For the savings application, you can check that plugging in $U = e^{-\rho t} u(c_t)$ and $f = w_t + r_t a_t - c_t$ will yield the Hamiltonian optimality conditions (1) stated above (but in present value terms, not current). For our more general W, the necessary conditions are

$$\begin{aligned}0 &= H_c(t, a_t, c_t, \lambda_t) \\ -\dot{\lambda}_t &= H_a(t, a_t, c_t, \lambda_t)\end{aligned}$$

$$\dot{a}_t = H_\lambda(t, a_t, c_t, \lambda_t).$$

Clearly, the same variational argument can be used from the start since

$$J(s, a; \{c_t\}) = \int_s^T \{H(t, a_t, c_t, \lambda_t) - \dot{a}_t]\}\, dt + Z(T, a_T),$$

which makes clear where the Hamiltonian is coming from. Next, we prove that the HJB equation admits the value function.

Theorem 1 (Verification Theorem) *Suppose there exists a function $V : \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ that is C^1 and satisfies*

$$U(t, a, c) + V_t(t, a) + V_a(t, a) \cdot f(t, a, c) \leq 0$$

for all feasible (t, a, c), and $V(t, a) = G(t, a)$ for all $(t, a) \in C$. Then $V(t, a) = W(t, a)$. If there further exists functions (a_t, c_t) that satisfy the law of motion and the HJB equation for all t, and lands in the target set C at time T. Then $\{c_t\}$ is an optimal control.

Proof: All the theorem is laying out is to make sure that the HJB is integrable. For any feasible c_t, integrating from s to T we obtain:

$$\begin{aligned}
0 \geq & \int_s^T U(t, a_t, c_t) dt + \int_s^T [V_t(t, a_t) + V_a(t, a_t)\dot{a}_t]\, dt \\
= & \int_s^T U(t, a_t, c_t) dt + \int_s^T [dV(t, a_t)/dt]\, dt \\
= & \int_s^T U(t, a_t, c_t) dt + V(T, a_T) - V(s, a) \\
= & \int_s^T U(t, a_t, c_t) dt + Z(T, a_T) - V(s, a) \\
= & J(s, a, \{c_t\}) - V(s, a)
\end{aligned}$$

and since this holds for all feasible (a_t, c_t), V must be the optimum. □

Note that just like when we *assumed* the value function exists to write down the Bellman equation in discrete time, we are *assuming* that it exists to derive the HJB equation. Whether it exists or not is a different question that we will not deal with.

2.2 Fixed Terminal Time/Free Terminal State

For most of our purposes, we want to think about $T \to \infty$. For completeness though, let us also think about optimal stopping time problems. First consider the case when T is still fixed, but we are free to choose a_T. All that changes from the variational argument above is that we now need to add a term[1]

$$\Delta_J \approx \text{same as above} + [Z_a(T, a_T) - \lambda_T]\, \Delta_a(T).$$

Hence we now additionally need that the costate satisfies

$$Z_a(T, a_T) = \lambda_T = V_a(T, a_T).$$

This is a form of a *smooth-pasting condition*: Not only must the value function hit the scrap value at time T (*value-matching condition*), but also it's derivative must be equalized. This is an example in which continuous time solutions turn out to be smooth—in disrete time, we would have a jump at T because we would only have inequality constraints between times $T - 1$ and T.

2.3 Free Terminal Time and State

When $T < \infty$, this is the full optimal stopping time problem. Not much changes again, except now we also need to consider the variational impact coming from T on the state and control (a_t, c_t) functions we want to choose. Assuming our HJB $V(t, a)$ exists for a given T, we want to find our function λ_t that additionally guarantees

$$\begin{aligned}\frac{dV(s, a; T)}{dT} &= \int_s^T \left[\mathcal{H}_a + \dot{\lambda}\right] \cdot (da_t/dT)dt + \int_s^T \mathcal{H}_c \cdot (dc_t/dT)dt \\ &\quad + \mathcal{H}(T, a_T, c_T, \lambda_T) + Z_t(T, a_T) + [Z_a(T, a_T) - \lambda_T]\,(da_T/dT).\end{aligned}$$

[1]There is still no need to worry about $\lambda_s a_s$, since $a_s = a$ is given.

Since all other terms would already be zero by the above arguments made above, the optimal stopping time T is determined by

$$0 = U(T, a_T, c_T) + \lambda_T \cdot f(T, a_T, c_T) + Z_a(T, a_T),$$

where $V_a(T, a_T) = \lambda_T$ yields the second smooth-pasting condition on the value function.

Note that we never need to worry about the choice of λ_s as long as a_s is given, since it is implied by the terminal choice. What the costate is trying to do, in addition to facilitating the choice of a_t, is find N additional boundary conditions for the state a_t (when there are $2N$ differential equations, you need $2N$ boundary conditions). When both terminal time/state are fixed, there was no need to; when the terminal state is a choice we need to; and when T is a choice we need to know *when* to apply the boundary. The terminal condition is also called the transversality condition (TVC).

2.4 Infinite Terminal Time/Free Terminal State

Finally, we arrive at the case when time T is infinite. One would think that it should be some condition on $\lim_{T\to\infty} \lambda_T a_T$; but since there is no "terminal" value, it is not clear what this value should be. In most economic problems, it will turn out that this value must converge to zero. However, this is not general, and there can are non-trivial examples in which the TVC is a different condition altogether. Nonetheless, we will focus on these type of problems.

Why we do so is because most economic problems come with a notion of discounting. To this end, suppose that

$$U(t, a, c) = e^{-\rho t} u(t, a, c) \quad \Rightarrow \quad \mathcal{H}(t, a, c, \lambda) = e^{-\rho t} u(t, a, c) + \lambda f(t, a, c).$$

Define a new costate $\hat{\lambda}_t \equiv e^{\rho t} \lambda_t$, and the current value Hamiltonian as

$$\hat{\mathcal{H}}(t, a, c, \lambda) = e^{\rho t} \mathcal{H}(t, a, c, \lambda) = u(t, a, c) + \hat{\lambda} f(t, a, c).$$

Then since

$$\dot{\hat{\lambda}}_t = \rho \hat{\lambda}_t + e^{\rho t} \dot{\lambda}_t,$$

the optimality conditons become

$$\begin{aligned} U_c(t, a_t, c_t) &= \lambda_t && \Rightarrow \quad u_c(t, a_t, c_t) = \hat{\lambda}_t, \\ -\dot{\lambda}_t &= e^{-\rho t} u(t, a_t, c_t) + \lambda_t \cdot f(t, a_t, c_t) && \Rightarrow \quad -\dot{\hat{\lambda}}_t + \rho \hat{\lambda}_t = u(t, a_t, c_t) + \hat{\lambda}_t f(t, a_t, c_t), \\ \dot{a}_t &= f(t, a_t, c_t), \end{aligned}$$

which now you can verify are the optimality conditions in (1), which we had expressed in current value form. For completeness, the HJB equation and Hamiltonian dynamics with discounting is

$$\rho V(t, a) = V_t(t, a) + \max_c \{u(t, a, c) + V_a(t, a) \cdot f(t, a, c)\} = V_t(t, a) + \hat{\mathcal{H}}^*(t, a, V_a(t, a)),$$

which, again, can be interpreted as a no-arbitrage condition, subject to the law of motion

$$\dot{a}_t = f(t, a, c)$$

and the optimal control satisfies

$$\begin{aligned} \hat{\mathcal{H}}^*(t, a, V_a(t, a)) &\equiv \max_{\{c_t\}} \hat{\mathcal{H}}(t, a, c, V_a(t, a)) \\ 0 &= \hat{\mathcal{H}}_c(t, a, c, V_a(t, a)) \\ u_c(t, a, c) &= -V_a(t, a) f_c(t, a, c). \end{aligned}$$

The TVC we will assume in most cases is $\lim_{T \to \infty} e^{-\rho T} \hat{\lambda}_T a_T = 0$, which implies that the marginal value of the state cannot grow faster than the rate of time preference.