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1. The Roadmap

Before you jump into all this math, a few things to note:

1. Econ $\neq$ Math.
2. Unfortunately, you're going to have to know all this stuff at least for this course and the prelims. Following that,

- If you don't do macro, no one's really going to care whether you remember all this.
- Even if you do macro, no one's really going to care whether you remember all this. BUT...
  (a) If you do quantitative macro, it's nice to know whatever you tell your computer to do actually has some theory behind it. And you might want to cite some theorems just to let people know you're not stupid.
  (b) If you do macro theory, it's nice to know there's a bunch of theorems you can use even if you don't really remember why they're true.

To be a bit serious, it's terribly boring if you just go through the math for the sake of math. There's a reason we're doing this: consider the following dynamic programming problem

$$V(x) = \sup_y \left[F(x,y) + \beta V(y)\right]$$
$$s.t. \; y \text{ is feasible given } x,$$

where x is the beginning of period state variable, y is the control variable, and $F(x,y)$ is the current period return function. As we said in class, it's the value function V and the solution to the sup problem, $y^* = g(x)$, that we're going after. To do this, we proceed in the following steps.

1. Find conditions such that an iterative procedure will admit V. What we're going to do is create a sequence of guesses, and wish that it converges to the true function.
   (a) The *Contraction Mapping Theorem (CMT)* tells us conditions under which we get convergence.
   (b) To use the theorem, we need to define the space in which V lives, and a notion of convergence for this space.
   (c) *Blackwell's Sufficiency Conditions* characterize when the CMT applies to our space.
2. So we know how to get V and g. The *Theorem of the Maximum* characterizes what these guys will look like.

In short, that's the whole point about Stokey and Lucas (1989) Ch.3. Proving whether this value function is actually identical to the sequence problem is another issue, which is the subject of Ch.4.

2. Spaces and Sequences

DEFINITION 1 *A metric space is a set* S*, together with a metric* $\rho : S \times S \to R$*, such that for all* x, y*,* $z \in S$ *:*

1. $\rho(x, y) \geq 0$*, with equality if and only if* $x = y$
2. $\rho(x, y) = \rho(y, x)$ *and*
3. $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$

EXERCISE 1 STOKEY AND LUCAS (1989), 3.3C, P.45 *The set* S *of all continuous, strictly increasing functions on* $[a, b]$ *with* $\rho(x, y) = \max_{a \leq t \leq b} |x(t) - y(t)|$ *is a metric space.*

Proof:

$$
\begin{aligned}
(i) \quad & \rho(x,y) = \max_{a\leq t\leq b} |x(t) - y(t)| = |x(t^*) - y(t^*)| \geq 0, \quad \text{where } t^* \text{ is the maximizer,} \\
& \rho(x,y) = \max_{a\leq t\leq b} |x(t) - y(t)| = 0 \quad \text{iff} \quad x(t) = y(t)\, \forall t \in [a,b], \text{ i.e. } x = y \\
(ii) \quad & \rho(x,y) = \max_{a\leq t\leq b} |x(t) - y(t)| = \max_{a\leq t\leq b} |y(t) - x(t)| = \rho(y,x). \\
(iii) \quad & \rho(x,z) = \max_{a\leq t\leq b} |x(t) - z(t)| = |x(t^*) - z(t^*)| \\
& \qquad \leq |x(t^*) - y(t^*)| + |y(t^*) - z(t^*)| \\
& \qquad \leq \max_{a\leq t\leq b} |x(t) - y(t)| + \max_{a\leq t\leq b} |y(t) - z(t)| \\
& \qquad = \rho(x,y) + \rho(y,z), \quad \text{where } t^* \text{ is the maximizer.}
\end{aligned}
$$

□

DEFINITION 2 *A sequence* $\{x_n\}_{n=0}^{\infty}$ *in* S *converges to* $x \in S$*, if for each* $\epsilon > 0$*, there exists* N_ϵ *such that*

$\rho(x_n, x) < \epsilon$ *for all* $n \geq N_\epsilon$.

DEFINITION 3 *A sequence* $\{x_n\}_{n=0}^{\infty}$ *in* S *is a Cauchy sequence if for each* $\epsilon > 0$*, there exists* N_ϵ *such that*

$\rho(x_n, x_m) < \epsilon$ *for all* $n, m \geq N_\epsilon$.

EXERCISE 2 STOKEY AND LUCAS (1989), 3.5B, P.46 *If a sequence* $\{x_n\}$ *in a metric space* (S, ρ) *converges, it is Cauchy.*

Proof: Let $x_n \to x$; then for any $\epsilon > 0$, $\exists N_\epsilon$ s.t. $\rho(x_n, x) < \epsilon/2\ \forall n \geq N_\epsilon$. Hence for $m, n \geq N_\epsilon$,

$$\rho(x_n, x_m) \leq \rho(x_n, x) + \rho(x_m, x) \leq \epsilon,$$

so $\{x_n\}$ is Cauchy. □

DEFINITION 4 *A metric space* (S, ρ) *is complete if every Cauchy sequence in* S *converges to an element in* S.

EXERCISE 3 STOKEY AND LUCAS (1989), 3.6A, P.47 *The metric space (S, ρ) in Example 1 is not complete.*

Proof: Consider the following an example of a Cauchy sequence in S that converges to a point that is not in S,

$$\left\{ x_n(t) = \frac{t}{n} \right\}_{n=1}^{\infty} \text{ where } t \in [a, b].$$

Each element of this sequence is a continuous and strictly increasing on $[a, b]$. Hence, $\{x_n\}$ is contained in S. This is also a Cauchy sequence since

$$\begin{aligned} \rho(x, y) &= \max_{a \le t \le b} |x_n(t) - x_m(t)| = \max_{a \le t \le b} \left| \frac{t}{n} - \frac{t}{m} \right| \\ &= \left| \frac{1}{n} - \frac{1}{m} \right| \max_{a \le t \le b} |t| \le \left[\left| \frac{1}{n} \right| + \left| \frac{1}{m} \right| \right] \max_{a \le t \le b} |t|, \end{aligned}$$

which can be made arbitrarily small by picking n and m large enough. Limit of this sequence of functions, however, is not in S since $\lim_{n \to \infty} = 0$, which is not strictly increasing. Hence, not all Cauchy sequences in S converges to a limit in S, therefore the metric space (S, ρ) is not complete. □

THEOREM 1 STOKEY AND LUCAS (1989), P.47 *Let $X \subseteq \mathbb{R}^L$, and $C(X)$ be the space of bounded continuous functions $f : X \to \mathbb{R}$ with the sup-norm. Then $C(X)$ is complete.*

Proof: The theorem is proven in 3 steps: first, we find a candidate limit function f; second, we show that $||f_n - f|| \to 0$ as $n \to \infty$; and third, we establish that $f \in C(X)$.

i) Let $\{f_n\} \subset C(X)$ be Cauchy. Then for any $\epsilon > 0$, $\exists N_\epsilon$ s.t. $\forall m, n \ge N_\epsilon$, $||f_n - f_m|| < \epsilon$. Now fix arbitrary $x_0 \in X$. Then we have

$$|f_n(x_0) - f_m(x_0)| \le ||f_n - f_m|| < \epsilon \quad \forall m, n \ge N_\epsilon.$$

Hence the sequence of real numbers $\{f_n(x_0)\}$ is Cauchy, and therefore converges to a limit point by the completeness of $\mathbb{R}$. For each x_0, let the limit point of $f_n(x_0)$ be $f(x)$. This allows us to define a function $f : X \to \mathbb{R}$, which will be our candidate limit function for f_n.[1]

ii) Since $\{f_n\}$ is Cauchy, for any $\epsilon > 0$, $\exists N_\epsilon$ s.t. $\forall m, n \ge N_\epsilon$, $||f_n - f_m|| \le \frac{\epsilon}{2}$. Now fix arbitrary $x_0 \in X$. Then $\forall m \ge n \ge N_\epsilon$,

$$\begin{aligned} |f_n(x_0) - f(x_0)| &\le |f_n(x_0) - f_m(x_0)| + |f_m(x_0) - f(x_0)| \\ &\le ||f_n - f_m|| + |f_m(x_0) - f(x_0)| \\ &\le \epsilon/2 + |f_m(x_0) - f(x_0)|, \end{aligned}$$

and by part 1), for each x_0 we can choose arbitrarily large m separately so that $|f_m(x_0) - f(x_0)| < \epsilon/2$. Since x_0 was arbitrary, $||f_n - f|| \le \epsilon \; \forall n \ge N_\epsilon$.[2]

[1] Be careful to note the difference between $C(X)$ and $\mathbb{R}$.

[2] Note the difference between 1) and 2). In 1), we established that for each x_0, we can find N_{ϵ, x_0} s.t. $\forall n \ge N_{\epsilon, x_0}$,

iii) That f is bounded is obvious, since for each x_0, $f_n(x_0)$ converges in $\mathbb{R}$. For continuity, choose k s.t. $||f - f_k|| < \epsilon/3$ which is possible by part 2), and δ s.t. $||x - y|| < \delta$ implies $|f_k(x) - f_k(y)| < \epsilon/3$, which is possible since $f_k \in C(X)$. Then

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)| \\ &< 2||f - f_k|| + \epsilon/3 \\ &\leq \epsilon. \end{aligned}$$

□

Definition 5 *Let (S, ρ) be a metric space. A neighborhood of a point $p \in S$ is a set $N_r(p)$ consisting of all points $q \in S$ such that $\rho(p, q) < r$. The number r is called the radius of $N_r(p)$.*

Definition 6 *Let (S, ρ) be a metric space. A point p is a limit point of the set $E \subset S$ if every neighborhood of p contains a point $q \neq p$ such that $q \in E$.*

Definition 7 *Let (S, ρ) be a metric space. $E \subset S$ is closed if very limit point of E is a point of E.*

Theorem 2 Sundaram (1996), 1.20, p.22 *A set S in R^n is closed if and only if for all sequences $\{x_k\}$ such that $x_k \in S$ for all k and $x_k \to x$, it is the case that $x \in S$. If S is closed, any sequence formed by the elements of S can not escape from S.*

Definition 8 *Let (S, ρ) be a metric space. $E \subset S$ is bounded if there is a real number M and a point $q \in S$ such that $\rho(p, q) < M$ for all $p \in E$.*

Definition 9 *Let (S, ρ) be a metric space. $E \subset S$ is compact if every sequence in A has a subsequence convergent to a point of E.*

Theorem 3 *Let E be a compact set in a metric space. Then E is complete.*

Theorem 4 Rudin (1976), 2.34, p. 37 *Compact subsets of metric spaces are closed.*

Theorem 5 Rudin (1976), 2.35, p. 37 *Closed subsets of compact sets are compact.*

Theorem 6 Rudin (1976), 2.41, p. 40 *A set $S \subseteq R^n$ is compact if and only if it is closed and bounded.*

Theorem 7 (Weierstrass) *Suppose $D \subseteq R^n$ is compact, and $f : D \to R$ is continuous on D. Then, f attains a maximum and a minimum on D.*

Definition 10 *Let S and T be metric spaces, $E \subset S$, $p \in E$, and f maps E into T. Then f is said to be continuous at p if for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\rho_T(f(x),\ f(p)) < \epsilon \text{ for all points } x \in E \text{ for which } \rho_S(x, p) < \delta.$$

Definition 11 *Let $f : S \to T$ where $S \subseteq R^n$ and $T \subseteq R^l$. Then f is said to be continuous at $x \in S$ if*

$|f_n(x_0) - f(x_0)| < \epsilon$ for any $\epsilon > 0$. So N_{ϵ, x_0} can vary for each x_0. But in 2), we are establishing that for *any* x_0, we can find N_ϵ that does *not* depend on x_0 s.t. $\forall n \geq N_\epsilon$, $|f_n(x_0) - f(x_0)| < \epsilon$ for any $\epsilon > 0$.

for all sequences $\{x_k\}$ such that $x_k \in S$ for all k, and $x_k \to x$, it is the case that $\{f(x_k)\} \to f(x)$.

Theorem 8 Rudin (1976), 4.14, p. 89 *Suppose f is a continuous mapping of a compact metric space X into a compact metric space Y. Then $f(X)$ is compact.*

Definition 12 *If a sequence contains a convergent subsequence, the limit of the convergent subsequence is called a limit point of the original sequence.*

Definition 13 *The* $\limsup$ *of a real valued sequence is the supremum of the set of limit points, and the* $\liminf$ *of a real valued sequence is the infimum of the set of limit points. Given a sequence $\{x_k\}$, which might not necessarily converge and therefore might have many limit points,* $\limsup$ *and* $\liminf$ *gives the upper and lower bounds of the set of limit points.*

For example,

$$
\begin{aligned}
x_k &= \begin{cases} 1 & \text{if } k \text{ is odd} \\ \frac{k}{2} & \text{if } k \text{ is even} \end{cases} \\
&= \{1, 1, 1, 2, 1, 3, 4, 1.....\}
\end{aligned}
$$

where $\limsup x_k = \infty$ and $\liminf x_k = 1$. Note that lim sup and lim inf are themselves limit points of a sequence. Therefore, in order to show that a sequence converges we can use the following theorem.

Theorem 9 Sundaram (1996), 1.18, p.21 *A sequence in R converges to a limit point $x \in R$ if and only if* $\limsup x_k = \liminf x_k = x$.

3. Contraction Mapping Theorem

Lemma 1 *If (S, ρ) is a metric space and $T : S \to S$ is a contraction mapping with modulus β, T is continuous.*

Proof: We need to show that for any $\epsilon > 0$ and all $s_0 \in S$, $\exists \delta_{\epsilon,s_0}$ s.t.

$$\rho(s, s_0) < \delta_{\epsilon,s_0} \text{ implies } \rho(Ts, Ts_0) < \epsilon \quad \forall s \in S.$$

Let $\delta_{\epsilon,s_0} = \epsilon$. Then

$$\rho(Ts, Ts_0) \leq \beta\rho(s, s_0) < \beta\delta_{\epsilon,s_0} = \beta\epsilon < \epsilon.$$

□

Theorem 10 (Contraction Mapping) *If (S, ρ) is a complete metric space, and $T : S \to S$ is a contraction mapping with modulus β,*

i. T has a unique fixed point $v^* \in S$, and

ii. $\forall v_0 \in S$, $\rho(T^n v_0, v^*) \leq \beta^n \rho(v_0, v^*)$ $\forall n$.

Proof:

i) Choose arbitrary $v_0 \in S$, and define $\{v_n\}_{n=0}^{\infty}$ by $v_{n+1} = Tv_n$ so that $v_n = T^n v_0$. Then

$$\rho(v_{n+1}, v_n) = \rho(Tv_n, Tv_{n-1}) \leq \beta\rho(v_n, v_{n-1}) = \cdots \leq \beta^n \rho(v_1, v_0).$$

Hence for arbitrary $m > n$,

$$\begin{aligned}
\rho(v_m, v_n) &\leq \rho(v_m, v_{m-1}) + \cdots + \rho(v_{n+1}, v_n) \\
&\leq \rho(v_1, v_0)[\beta^{m-1} + \cdots + \beta^n] \\
&= \beta^n(\beta^{m-n-1} + \cdots + 1)\rho(v_1, v_0) \\
&\leq \frac{\beta^n}{1-\beta}\rho(v_1, v_0).
\end{aligned}$$

This implies that $\{v_n\}$ is Cauchy, and since S is complete, $\exists v^* \in S$ s.t. $v^* = \lim_{n\to\infty} v_n$. It is straightforward that v^* is a fixed point since

$$Tv^* = T(\lim_{n\to\infty} v_n) = \lim_{n\to\infty}(Tv_n) = \lim_{n\to\infty} v_{n+1} = v^*,$$

where the second equality is ensured by the continuity of T.

To establish uniqueness, suppose that $\exists \hat{v} \neq v^*$ s.t. $T\hat{v} = \hat{v}$. Then

$$0 < \rho(v^*, \hat{v}) = \rho(Tv^*, T\hat{v}) \leq \beta\rho(v^*, \hat{v}),$$

which is a contraction since $\beta \in (0, 1)$.

ii) Use induction. For $n = 0$, it is straightforward to check $\rho(T^0 v_0, v^*) \leq \beta^0 \rho(v_0, v^*)$. Now suppose $\rho(T^k v_0, v^*) \leq \beta^k \rho(v_0, v^*)$. Then

$$\rho(T^{k+1}v_0, v^*) = \rho[T(T^k v_0), Tv^*] \leq \beta\rho(T^k v_0, v^*) \leq \beta^{k+1}\rho(v_0, v^*).$$

□

THEOREM 11 BLACKWELL'S SUFFICIENCY CONDITIONS *Let $X \subseteq \mathbb{R}^L$, and $B(X)$ be the space of all bounded functions $f : X \to \mathbb{R}$ with the sup-norm $d(\cdot)$. Let $T : B(X) \to B(X)$ be an operator that satisfies*

i) monotonicity: if $f, g \in B(X)$ and $f(x) \leq g(x)\ \forall x \in X$, $(Tf)(x) \leq (Tg)(x)\ \forall x \in X$.

ii) discounting: $\exists \beta \in (0, 1)$ s.t.

$$T(f(x) + a) \leq (Tf)(x) + \beta a \quad \forall f \in B(X), a \geq 0, x \in X.$$

Then T is a contraction mapping with modulus β.

Proof: Write $f \leq g$ if $f(x) \leq g(x)\ \forall x \in$ X. For $f \leq g$,

$$\begin{aligned}
& f \leq g + d(f, g) \\
\Rightarrow\ & Tf \leq T[g + \rho(f, g)] \quad \text{(by monotonicity)}
\end{aligned}$$

$$\le Tg + \beta\rho(f,g) \quad \text{(by discounting)}$$
$$\Rightarrow |Tf - Tg| \le \beta\rho(f,g),$$

and since this holds $\forall x \in X$,

$$\rho(Tf, Tg) \le \beta\rho(f,g).$$

□

4. The Theorem of Maximum

DEFINITION 14 *A correspondence Γ from $X \subseteq R^l$ into $Y \subseteq R^m$ is a map which associates with each element $x \in X$, a (non-empty) subset $\Gamma(x) \subseteq Y$.*

Consider the following dynamic programming problem,

$$V(x) = \sup_y \left[F(x,y) + \beta V(y)\right]$$

$$s.t.\ y \text{ is feasible given } x,$$

where $x \in X \subseteq R^l$ is the beginning of period state variable, $y \in X$ is the end-of-period state variable (or control variable) to be chosen, and $F(x,y)$ is the current period return function. Correspondences are used to denote the relationship between the current state variable, x, and the choice variable, y. A *feasibility correspondence*, $\Gamma : X \to X$, is used to define which values of y are feasible given x. We would like to know how $\Gamma(x)$ behaves as x changes over X in order to be able to characterize how the maximizing values of y and value function $V(x)$ behaves over X. Hence, we need to introduce a notion of continuity for correspondences.

DEFINITION 15 *$\Gamma : X \to Y$ is a compact-valued correspondence if $\Gamma(x)$ is a compact set of Y for each $x \in X$.*

DEFINITION 16 *$\Gamma : X \to Y$ is a closed-valued correspondence if $\Gamma(x)$ is a closed set of Y for each $x \in X$.*

DEFINITION 17 *$\Gamma : X \to Y$ is a convex-valued correspondence if $\Gamma(x)$ is a convex set of Y for each $x \in X$.*

To give an example of a correspondence with these properties consider,

$$\Gamma(x) = \{y \mid 0 \le y \le x\}, \text{ where } X \subseteq R_+, \text{ and } Y \subseteq R_+.$$

DEFINITION 18 *Graph of a correspondence $\Gamma(x)$ is the set A defined as,*

$$A = \{(x,y) \mid y \in \Gamma(x)\}.$$

DEFINITION 19 *$\Gamma : X \to Y$ is a closed-graph correspondence if A is a closed set.*

DEFINITION 20 *$\Gamma : X \to Y$ is a convex-graph correspondence if A is a convex set.*

Note that a closed-graph correspondence is also closed-valued, and a convex-graph correspondence is also convex-valued. The converses, however, do not hold.

Lower Hemi-Continuity

Definition 21 *A correspondence $\Gamma : X \to Y$ is lower hemi-continuous (l.h.c.) at x, if $\Gamma(x)$ is non-empty and if, for every $y \in \Gamma(x)$ and every sequence $x_n \to x$, there exists $N \geq 1$ and a sequence $\{y_n\}_{n=N}^{\infty}$ such that $y_n \to y$ and $y_n \in \Gamma(x_n)$, $\forall n \geq N$.*

Note that in order to check l.h.c. of a correspondence at x, we first pick any point $y \in \Gamma(x)$ and a sequence $x_n \to x$. Then, we look for a sequence y_n which is contained in $\Gamma(x_n)$, and converges to the point y. Since we first pick any point y in the image of x, l.h.c. fails if there exists a sudden "blow-up" in the correspondence.

Upper Hemi-Continuity

Definition 22 *A compact valued correspondence $\Gamma : X \to Y$ is upper hemi-continuous (u.h.c) at x if $\Gamma(x)$ is non-empty and if, for every sequence $x_n \to x$ and every sequence $\{y_n\}$ such that $y_n \in \Gamma(x_n)$ for all n, there exists a convergent subsequence of $\{y_n\}$ whose limit point y is in $\Gamma(x)$.*

Note that in order to check u.h.c. of a correspondence at x, we first pick $x_n \to x$ and a sequence y_n contained in the images of x_n. Then, we look for a convergent subsequence of y_n which converges to a point y in the image of x. Upper hemi-continuity will fail if there is a sudden "collapse" in the correspondence. Then, we could pick x_n and y_n, but fail to find a point y in the image of x such that a subsequence of y converges to that point.

Definition 23 *A correspondence $\Gamma : X \to Y$ is continuous at $x \in X$ if it is both u.h.c. and l.h.c.*

Now consider the following optimization problem,

$$\sup_{y \in \Gamma(x)} f(x, y),$$

where $f : X \times Y \to R$ is a single valued function, and $\Gamma : X \to Y$ is a non-empty correspondence. Define the *maximized value function* $h(x)$ as,

$$h(x) = \max_{y \in \Gamma(x)} f(x, y),$$

and the *set of maximizers* $G(x)$ as,

$$G(x) = \{y \in \Gamma(x) \mid f(x, y) = h(x)\}.$$

Theorem 12 (Theorem of the Maximum) *Let $X \subseteq \mathbb{R}^L$, $Y \subseteq \mathbb{R}^M$, $f : X \times Y \to \mathbb{R}$ be a continuous function, and $\Gamma : X \to Y$ be a compact-valued and continuous correspondence. Then*

i) the function $h : X \to \mathbb{R}$ defined as $h(x) = \max_{y \in \Gamma(x)} f(x, y)$ is continuous, and *ii) the correspondence $G : X \to Y$ defined as $G(x) = \{y \in \Gamma(x) : h(x) = f(x,y)\}$ is nonempty, compact-valued and upper-hemicontinuous.*

Proof: a) shows that $G(x)$ is nonempty and compact-valued and b) shows that $G(x)$ is u.h.c. This establishes ii). c) proves i).

a) Fix $x \in X$. The set $\Gamma(x)$ is nonempty (implied by continuity) and compact, and $f(x,\cdot)$ is continuous; hence $\max_{y\in\Gamma(x)}$ is attained and $G(x)$ is nonempty by the Weierstrass Theorem. It is clear that G(x) is bounded since $G(x) \subseteq \Gamma(x)$ and $\Gamma(x)$ is bounded. So to show that $G(x)$ is compact, it remains to show that it is closed.
Suppose $y_n \to y$ and $y_n \in G(x)\ \forall n$. Since $\Gamma(x)$ is compact which implies it is closed, $y \in \Gamma(x)$. Since $h(x) = f(x, y_n)\ \forall n$, and f is continuous,

$$h(x) = \lim_{n\to\infty} f(x, y_n) = f(x, \lim_{n\to\infty} y_n) = f(x,y).$$

Hence $y \in G(x)$, so $G(x)$ is closed, and hence compact.

b) Fix x, and let $\{x_n\} \subseteq X$ be any sequence converging to x. Choose $y_n \in G(x) \subseteq \Gamma(x)\ \forall n$ and arbitrary $z \in \Gamma(x)$. Then

since Γ is u.h.c., $\exists\{y_{n_k}\} \subseteq \{y_n\}$ s.t. $y_{n_k} \to y \in \Gamma(x)$ and
since Γ is l.h.c., $\exists\{z_{n_k}\}$ s.t. $z_{n_k} \to z$ and $z_{n_k} \in \Gamma(x_{n_k}),\quad \forall k$.

Now we have $f(x_{n_k}, y_{n_k}) \geq f(x_{n_k}, z_{n_k})\ \forall k$ since $y_{n_k} \in G(x_n)$, and hence

$$\begin{array}{rccc} & \lim_{k\to\infty} f(x_{n_k}, y_{n_k}) & \geq & \lim_{k\to\infty} f(x_{n_k}, z_{n_k}) \\ \Rightarrow & f(\lim_{k\to\infty} x_{n_k}, \lim_{k\to\infty} y_{n_k}) & \geq & f(\lim_{k\to\infty} x_{n_k}, \lim_{k\to\infty} z_{n_k}) \\ \Rightarrow & f(x,y) & \geq & f(x,z) \end{array}$$

by the continuity of f. Since $z \in \Gamma(x)$ was arbitrary, $y \in G(x)$, and hence G is u.h.c.

c) Fix x, and let $\{x_n\} \subseteq X$ be any sequence converging to x. Choose $y_n \in G(x_n)\ \forall n$. Let $\overline{h} = \limsup h(x_n)$ and $\underline{h} = \liminf h(x_n)$. Then there exists a subsequence $\{x_{n_k}\}$ s.t. $\overline{h} = \lim f(x_{n_k}, y_{n_k})$. Since G is u.h.c., there exists a subsequence of $\{y_{n_k}\}$, say $\{y_{n_{k_j}}\}$, s.t. $y_{n_{k_j}} \to y \in G(x)$. Hence $\overline{h} = \lim f(x_{n_{k_j}}, y_{n_{k_j}}) = f(x,y) = h(x)$. Similarly, $h(x) = \underline{h}$. Hence $\lim h(x_n) = h(x) = h(\lim x_n)$, establishing continuity of h.

□

COROLLARY 1 *If Γ is compact-valued, continuous, and convex-valued; and f is continuous and strictly concave in y, then G is single-valued, therefore it is a continuous function, called g.*

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