Continuous Time Bewley Models

DEEQA Quantitative Macro

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Today

- Aiyagari with Poisson wage process
  - Based on `http://www.princeton.edu/~moll/HACT.pdf`, so won't go through all the details
- Interest rate follows geometric Brownian motion
  - Based on Angeletos (2007); Benhabib et al. (2014), extended in above paper

Income Fluctuation Problem in Continuous Time

- Continuum of agents solve

$$V(t,a,w) = \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt$$

$$da_t = (w_t + r_t a_t - c_t)dt, \quad a_t \geq -B,$$

- w_t is CPP with jumps z_w that arrive at rate λ_w:

  $z_w = \gamma_2 - \gamma_1$ and $\lambda_w = \lambda_1$ if $w_t = \gamma_1$

  $z_w = \gamma_1 - \gamma_2$ and $\lambda_w = \lambda_2$ if $w_t = \gamma_2$

Income Fluctuation Problem in Continuous Time

- With stationarity, $r_t = r$. Since w takes only two values, let's instead write

$$V(a,i) = \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt$$
$$da = (ra + \gamma_i - c)\, dt + (a - a^-) dN_{it}$$

  where $i = 1, 2$ and N_{it} is Poisson with rate λ_i.
- Thus we can take a_t as the CPP instead of w_t

Income Fluctuation Problem in Continuous Time

- With stationarity, $V_t = 0$. Let $j = 3 - i$. The HJB equation is

$$\rho V(a,i) = \max_c \{u(c) + V_a \cdot [ra + \gamma_i - c]\} \\ + \lambda_i [V(a,j) - V(a,i)]$$

- Lastly, need boundary condition

$$u'(\gamma_i - rB) \geq V_a(-B, i)$$

which is just the Kuhn-Tucker condition at the boundary.

Some Theoretical Results

Suppose $r < \rho$ and $\gamma_1 < \gamma_2$.

1. At the constraint, $s_1(-B) = 0$ but $s_1(a) < 0$ for all $a > -B$:
   Agents in bad state always borrow when they can
2. Suppose DARA. Then there exists a_M s.t.
   $s_2(a) \approx -(a - a_M)$ for all $a \geq a_M$.
   Stationary equilibrium is bounded when $r < \rho$
3. Easy to show savings explodes if $r > \rho$

Stationary Distribution in Continuous Time

- The solution to the Hamiltonian satisfies $u'(c(a,i)) = V_a$, so

$$da = (ra + \gamma_i - c(a,i))\,dt + (a - a^-)dN_{it}$$

- Let $p(t,a)$ denote the density of a at time t. With stationarity, $p_t = 0$ and we can ignore the time subscript.
- The KFE is thus

$$\frac{\partial}{\partial a}\underbrace{[ra + \gamma_i - c(a,i)]}_{\equiv s(a,i)}p(a) = \lambda_j p(a; j \neq i) - \lambda_i p(a)$$

Stationary Distribution in Continuous Time

- By definition, $p(a)$ is the distribution of a when state is i.[1]
- Hence we have the "set" of KFE's

$$\frac{d}{da}s_i(a)p_i(a) = \lambda_j p_j(a) - \lambda_i p_i(a), \quad i = 1, 2 \tag{1}$$

$$\Rightarrow \quad \frac{d}{da}[s_1(a)p_1(a) + s_2(a)p_2(a)] = 0,$$

- For $p(a)$ to have bounded support, the integrand

$$s_1(a)p_1(a) + s_2(a)p_2(a) = 0. \tag{2}$$

- Easy to solve the ODE resulting from plugging (2) in (1)

[1] Does NOT mean that a^- was in state j.

More Theoretical Results

- If $B <$NBL, Dirac mass at B. If B is NBL, zero mass at B: Clustering at BL vs smooth wealth distribution
- For "small" λ_2, Dirac mass at a_M:
  "less" wealth inequality (short tail)
  For large λ_2, zero mass at a_M:
  more wealth inequality (long tail)

$\Rightarrow$ We already knew this, but means for large wealth inequality λ_2 must be high

- But intuitively, γ_2 must also be high relative to γ_1

$\Rightarrow$ save a lot *while* rich in anticipation of becoming poor

Investment Risk in Continuous Time

- Continuum of agents solve

$$V(t,k,b) = \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt$$
$$d(k_t + b_t) = (w + R_t k_t + r_t b_t - c_t) dt$$
$$dR_t = R dt + \sigma dW_t,$$

  where W_t is BM.
- Assume no wage risk; easy to incorporate if Poisson
- For simplicity, assume $k \geq 0$ but no borrowing limit
  Will think of constraints later

* Assume $u(c) = c^{1-\gamma}/(1-\gamma)$: Homotheticity!
  Makes policy rules linear

Income Fluctuation Problem in Continuous Time

- Total net worth $A_t \equiv w/r + k_t + b_t$ is stochastic with

$$dA_t = [rA_t + (R - r)\kappa A_t - c]\, dt + \sigma\kappa A_t dW_t$$

  where $\kappa \equiv k/A$ is the share invested in risky assets.
- Thus A_t is geometric BM
- HJB equation with Ito is

$$\rho V(A) = \max_{c,\kappa} \Big\{ u(c) + V_A \cdot [rA + (R - r)\kappa A - c] + \frac{\sigma^2}{2} \cdot V_{AA}\kappa^2 A^2 \Big\}$$

  Note that HJB includes Ito correction term *inside* the max operator because of κ

Some Theoretical Results

- From Merton (1969), if κ allowed to be arbitrarily large we obtain:

$$\kappa = \frac{R - r}{\gamma \sigma^2}$$

$$S = 1 - r + \frac{r - \rho}{\gamma} + \frac{(1-\gamma)(R-r)}{2\gamma}\kappa$$

  where $(1 - S) \equiv c/A$.
- Invest more in risky asset if return is high and risk is low
- Also save more when return is high and risk is low
- But note that investment and savings propensity is independent of wealth (κ, S are constants)

Stationary Distribution in Continuous Time

- Given the solution $c(A) = (1 - S)A$, the A_t process is

$$dA_t = \underbrace{\left[\frac{r-\rho}{\gamma} + \frac{(1+\gamma)(R-r)^2}{2\gamma^2\sigma^2}\right] A_t}_{\equiv s \cdot A} dt + \sigma\kappa A_t dW_t$$

  where s is a constant
- The KFE (for stationary distribution) is thus

$$s \cdot \frac{d\left[Ap(A)\right]}{dA} = \frac{\sigma^2\kappa^2}{2} \cdot \frac{d^2\left[A^2 p(A)\right]}{dA^2}$$

More Theoretical Results

- The KFE (for stationary distribution) is thus

$$\left[\frac{r-\rho}{\gamma} + \frac{(1+\gamma)(R-r)}{2\gamma\kappa}\right] \cdot \frac{d\left[Ap(A)\right]}{dA} = \frac{(R-r)}{2\gamma\kappa} \cdot \frac{d^2\left[A^2 p(A)\right]}{dA^2}$$

- Can easily guess $p(A) = \zeta A^{-\eta-1}$: distribution is Pareto

$$\eta = \frac{2(\rho - r)}{R-r} \cdot \frac{1}{\kappa} - \gamma$$

- Must be larger than 0 for Pareto to be defined (risk and discounting high compared to excess return)

Wealth Inequality

$$\eta = \frac{2(\rho - r)}{(R - r)} \cdot \frac{1}{\kappa} - \gamma = \gamma \left[\frac{2\sigma^2(\rho - r)}{(R - r)^2} - 1 \right]$$

More wealth inequality (thicker Pareto tail) when

- More investment; higher excess return
- lower risk and more patience
- ...and less aversion? NO: higher EIS

...ALL THIS ONLY VALID IN PARTIAL EQUILIBRIUM

Some Caveats

- Since Pareto only works with $A > 0$, only applies for large levels of wealth
- Need to add borrowing constraint for lower levels, but then need some modifications to the problem
- Top will still be approximately Pareto

General Equilibrium

- Since policy rules are linear, easily aggregated
- Assume $L = 1$ inelastic; representative firm with technology $F(K, L) = f(K)$ and depreciation rate δ:

$$R = f'(K) - \delta, \quad w = f(K) - Kf'(K)$$

- K and r determined by

$$K = \frac{w\kappa}{r(1-\kappa)}, \quad (1-\kappa)r + \kappa R = 1 - S$$

Complete vs. Incomplete Markets

- With more risk, R down: tomorrow is more expensive
- If γ low (high EIS), substitution effect dominates
  $\Rightarrow$ invest less
- if κ high, income effect dominates
  $\Rightarrow$ invest more
- Angeletos (2007) shows there exists γ low enough so that SS capital is lower

Angeletos, George-Marios, "Uninsured idiosyncratic investment risk and aggregate saving," *Review of Economic Dynamics*, 2007, *10* (1), 1 – 30.

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Merton, Robert C, "Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case," *The Review of Economics and Statistics*, August 1969, *51* (3), 247–57.