There are 3 big questions for a total of 25+25+20=70 points. Each big question is split into several small questions. Allocate your time wisely.

Questions

1. *(Jumps in Researchers, 25 points).* This question is based on the Solow technology growth model with no intermediate good sector. Remember we defined μ as the share of labor devoted to research simple model of technology growth:

$$Y = K^{\alpha}(ZL_Y)^{1-\alpha}$$
$$\dot{K} = sY - \delta K, \quad L_Y = (1-\mu)L$$
$$\dot{Z} = \eta L_Z^{\lambda} Z^{\phi}, \quad L_Z = \mu L$$

Assume that $\phi, \lambda \in (0,1)$. Suppose the economy is at a BGP with $\mu = \mu_0$, where technology growth was constant at g_0.

(a) *(4 points).* First, normalize all variables into effective units. Ignore idea production for now, and derive an expression for per capita output on a BGP assuming g_0 is constant.

$$\hat{y} = (1-\mu)^{1-\alpha}\hat{k}^{\alpha}, \qquad \dot{\hat{k}} = s\hat{y} - (n + g_0 + \delta)\hat{k}$$

Since $\dot{\hat{k}} = 0$ *on a BGP,*

$$\hat{k} = \left(\frac{s}{n+g_0+\delta}\right)^{\frac{1}{1-\alpha}}(1-\mu), \qquad y = \left(\frac{s}{n+g_0+\delta}\right)^{\frac{\alpha}{1-\alpha}}(1-\mu)Z$$

(b) *(4 points).* Derive expressions for g_0 on the original BGP.

$$g_0 = \eta L_Z^{\lambda} Z^{\phi-1} \quad \Rightarrow \quad 0 = \lambda n - (\phi - 1)g_0 \quad \Rightarrow \quad g_0 = \frac{\lambda n}{1-\phi}$$

Now, suddenly μ increases to $\mu_1 > \mu_0$.

(c) *(4 points).* What is the level of technology growth g_1 once we reach the new BGP?
Same as before.

(d) *(4 points).* Demonstrate on a graph, with time on the x-axis and $\dot{Z}/Z$ on the y-axis, how the growth rate of technology evolves over time.
Growth rate is constant, jumps vertically, and slowly converges back.

(e) *(4 points).* Demonstrate on a graph, with time on the x-axis and $\log Z$ on the y-axis, how technology increases over time.
Increases linearly, curves up concavely, and becomes linear again with same slope. But there is a level shift.

(f) *(5 points).* Demonstrate on a graph, with time on the x-axis and $\log(Y/L)$ on the y-axis, how output per worker increases over time. Do we know whether GDP per worker is higher or lower than if μ had not changed in the long-run?
Initially jumps down, then curves up concavely. However, we do not know whether there is a level shift, and even if there is a shift, whether it goes up or down.

2. *(RCK Model with Taxes, 25 points).* Consider the following simple equilibrium version of the RCK model we learned in class, and you solved in your homework. A representative household supplies labor inelastically and solves

$$\max_{c,a} U(c) = \int_0^\infty e^{-\rho t} u(c_t) dt$$
$$\text{s.t.} \quad \dot{A}_t = (1-\tau) w_t L_t + (1-\tau) r_t A_t - C_t,$$
$$A_0 > 0 \text{ given,}$$

where

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}.$$

Remember that c_t is per capita consumption, while C_t is aggregate consumption.

A government levies taxes τ on both wages and interest income, and uses it to finance government expenses G. G is *not* redistributed to households, but completely wasteful from the viewpoint of the households:

$$\tau(w_t L_t + r_t A_t) = G. \tag{1}$$

Note that (τ, G) are not indexed by t; that is, government policies are assumed to be time-invariant.

A representative firm solves

$$\max_{K_t, L_t} \left\{ K_t^\alpha (Z_t L_t)^{1-\alpha} - R_t K_t - w_t L_t \right\}.$$

The population L_t grows at rate n, and technology Z_t at rate g. In equilibrium, the rental rate of capital satisfies $R_t = r_t + \delta$.

(a) *(4 points).* Write down the current-value Hamiltonian of the representative household, $\mathcal{H}(\hat{c}_t, \hat{k}_t, \lambda_t)$, where λ_t is the costate. Normalize all variables into "effective" units: use lowercase variables to express per capita units, and "hats" for the effective units. Be clear which variables have to be normalized.

$$\mathcal{H}(\hat{c}_t, \hat{k}_t, \lambda_t) = u(\hat{c}_t) + \lambda_t \{(1-\tau) w_t + [(1-\tau) R_t - n - g - (1-\tau)\delta] \hat{a}_t - \hat{c}_t\}$$

(b) *(4 points).* Derive the optimality conditions for the representative household.

$$u'(\hat{c}_t) = \lambda_t$$
$$-\dot{\lambda}_t = \lambda_t [(1-\tau) R_t - n - g - (1-\tau)\delta - \hat{\rho}]$$
$$\dot{\hat{a}}_t = (1-\tau) w_t + [(1-\tau) R_t - n - g - (1-\tau)\delta] \hat{a}_t - \hat{c}_t$$

where $\hat{\rho} = \rho - g(1-\gamma)$.

(c) *(4 points).* Now, write down the first-order conditions for the representative firm. Transform the conditions into the effective units of K_t.

$$\alpha (Z_t L_t / K_t)^{1-\alpha} = R_t, \qquad (1-\alpha) Z_t (K_t / Z_t L_t)^\alpha = w_t$$

$$\alpha\hat{k}_t^{\alpha-1} = R_t, \qquad (1-\alpha)\hat{k}_t^{\alpha} = w_t/Z_t$$

(d) *(5 points).* Plug in the market clearing conditions $K_t = A_t$, as well as the budget balance condition (1) into the household's optimality conditions. What is the BGP level of *effective* capital per capita? Is it larger or smaller with or without taxes?

$$\begin{aligned} u'(\hat{c}_t) &= \lambda_t \\ -\dot{\lambda}_t &= \lambda_t \left[(1-\tau)\alpha\hat{k}_t^{\alpha-1} - n - g - (1-\tau)\delta - \hat{\rho}\right] \\ \dot{\hat{k}}_t &= (1-\tau)\hat{k}_t^{\alpha} - (n+g+(1-\tau)\delta)\hat{k}_t - \hat{c}_t \\ \Rightarrow \hat{k}^* &= \left[\frac{(1-\tau)\alpha}{n+g+(1-\tau)\delta+\hat{\rho}}\right]^{\frac{1}{1-\alpha}} = \left[\frac{\alpha}{\delta+(n+g+\hat{\rho})/(1-\tau)}\right]^{\frac{1}{1-\alpha}} \end{aligned}$$

Clearly, it is smaller, as the government takes away some of the resources. From a phase diagram, we can deduce that $\hat{c}^*$ is also smaller.

(e) *(4 points).* Derive the 2nd-order ODE that describes local dynamics around the BGP $(\hat{c}^*, \hat{k}^*)$, by approximating by Taylor expansion:

$$\ddot{\hat{k}}_t - \hat{\rho}\dot{\hat{k}}_t - \sigma(\tau)\hat{k}_t = -\sigma(\tau)\hat{k}^*,$$

where

$$\sigma(\tau) = \frac{(1-\tau)\alpha(1-\alpha)}{\gamma} \cdot \left[\hat{k}^{*\alpha-2}\right] > 0$$

and $\hat{k}^*$ is the constant, steady state level of $\hat{k}_t$.
I had a typo in the exam, where I omitted the $\hat{c}^*$ in

$$\sigma(\tau) = \frac{(1-\tau)\alpha(1-\alpha)}{\gamma} \cdot \left[\hat{k}^{*\alpha-2}\right] \hat{c}^* > 0.$$

Of course, no points will be taken off for missing the $\hat{c}^*$, and I will be extra-generous with the grading especially related to this question.

$$\begin{aligned} \dot{\hat{c}}_t &= \frac{1}{\gamma}\left[(1-\tau)\alpha\hat{k}_t^{\alpha-1} - n - (1-\tau)\delta - g - \hat{\rho}\right]\hat{c}_t \\ \dot{\hat{k}}_t &= (1-\tau)\hat{k}_t^{\alpha} - (n+g+(1-\tau)\delta)\hat{k}_t - \hat{c}_t \end{aligned}$$

$$\begin{aligned} \dot{\hat{c}}_t &\equiv \frac{(1-\tau)\alpha(\alpha-1)}{\gamma} \cdot \left[\hat{k}^{*\alpha-2}\right] \hat{c}^*(\hat{k}_t - \hat{k}^*) \\ \dot{\hat{k}}_t &= -(c_t - \hat{c}^*) + \hat{\rho}(\hat{k}_t - \hat{k}^*) \\ \Rightarrow \quad \ddot{\hat{k}}_t &= \underbrace{\frac{(1-\tau)\alpha(1-\alpha)}{\gamma} \cdot \left[\hat{k}^{*\alpha-2}\right] \hat{c}^*}_{\sigma(\tau)>0}(\hat{k}_t - \hat{k}^*) + \hat{\rho}\dot{\hat{k}}_t \\ \Rightarrow \quad \ddot{\hat{k}}_t - \hat{\rho}\dot{\hat{k}}_t - \sigma(\tau)\hat{k}_t &= -\sigma(\tau)\hat{k}^* \end{aligned}$$

The particular solution is clearly $\hat{k}^*$. The general solution is $\hat{k}_t = (k_0 - \hat{k}^*)e^{\alpha t}$, with

$$\alpha = \frac{\hat{\rho}}{2} - \sqrt{\hat{\rho}^2/4 + \sigma(\tau)} < 0$$

(f) *(4 points).* What happens to $(\hat{c}^*, \hat{k}^*)$ and the speed of convergence as τ grows larger?

The expression is negative and decreasing in σ, so the absolute value is increasing in σ. Since σ is decreasing in τ, the speed of convergence is also decreasing in τ. So not only do taxes decrease BGP output, but also it takes longer to get there.

3. *(Romer Model with International Capital Flows, 20 points).* Consider the following, very minor modification of the Romer model we learned in class. Assume that the interest rate is fixed at an international level, which the entire country takes as given. To simplify the algebra, assume that $\delta = 0$. **Then $R = r$ is taken as a parameter by both the consumers and producers.** Henceforth I only use r.

Otherwise the model is identical to what we learned in class: population does not grow, and normalize $L = 1$. A representative consumer solves

$$\max_{C,A} U(C) = \int_0^\infty e^{-\rho t} u(C_t) dt$$
$$\text{s.t.} \quad \dot{A}_t = w_t L + r_t A_t - C_t, \qquad A_0 > 0 \text{ given,}$$

where

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}.$$

Ideas are produced according to:

$$\dot{Z} = \eta L_Z Z,$$

where L_Z is the amount labor used in idea production. Final goods are produced according to:

$$Y = L_Y^{1-\alpha} \int_0^Z x_j^\alpha dj.$$

where L_Y is the amount of labor used in final good production.

(a) *(4 points).* Set up the final good producer's problem and write down its first-order conditions. Show that the price elasticity of p_j w.r.t. demand x_j is equal to $\alpha - 1$.

$$w = (1-\alpha) \cdot Y$$
$$p_j = \alpha \cdot \left[L_Y / x_j\right]^{1-\alpha}. \tag{2}$$

Take logs of the second equation and differentiate w.r.t. $\log x_j$ to get $\alpha - 1$:

$$d \log p_j(x_j) / d \log x_j = p'(x_j) x_j / p(x_j) = \alpha - 1$$

The profit maximization problem for intermediate good producers is

$$\max_{x_j} \left\{ p_j(x_j) x_j - r x_j \right\}$$

(b) *(3 points).* What price does the intermediate good producer set? (Derive it, don't just write down a memorized answer.)

Take f.o.c. w.r.t. x_j:

$$p_j'(x_j)x_j + p_j(x_j) = r \quad \Rightarrow \quad \alpha = r/p_j \quad \Rightarrow \quad p_j = r/\alpha.$$

(c) *(4 points).* Show that profits of a single producer can be written as

$$\pi = \frac{\alpha(1-\alpha)Y}{Z}.$$

Intermediate profits are

$$\pi = (p-r)x_j = \frac{1-\alpha}{\alpha} \cdot \frac{rK}{Z}$$

since $x_j = x = K/Z$, and from (2),

$$\begin{aligned} pZx &= \alpha \cdot (Zx)^{\alpha}(ZL_Y)^{1-\alpha} = \alpha Y \\ rK &= \alpha^2 Y \end{aligned}$$

and plugging this back in we get the desired expression.

(d) *(4 points).* The price of an idea is the discounted sum of future profits, $P_Z = \pi/r$. Set up the profit maximization problem of the research firm, and derive the first-order condition for idea production. Then, given the expression for w from (a), and π from (b), derive an expression for μ. How does the share of researchers change with the interest rate?

$$\begin{aligned} &\max_{L_Z} \left\{ P_Z \cdot \dot{Z} - wL_Z = \eta P_Z L_Z Z - wL_Z = (\eta P_Z Z - w)L_Z \right\} \\ &\eta P_Z Z = w \\ &\eta\alpha(1-\mu) = r \\ &\mu = 1 - r/\eta\alpha \end{aligned}$$

So higher r leads to lower research share.

(e) *(5 points).* We know that on a BGP, the representative consumer's optimality condition implies that $r = \gamma g + \rho$. Since r is fixed internationally, this means that

$$g^* = \frac{r-\rho}{\gamma}.$$

So when interest rates are high, the growth rate is high. Explain why the growth rate and share of researchers may not move together in this economy. What is going on here? Is all the raw capital used by the intermediate good producers supplied domestically?

When r is high, the consumer has more incentive to save, so growth is higher. But when r is high, the price of an idea is also lower, so there is less incentive to innovate. With international capital flows, intermediate good producers can use capital from abroad, and households can supply capital to intermediate good producers abroad. So the growth rate is high when interest rates are low because of higher capital, despite the country's own research and technology being low and not being used in production.