Ramsey-Cass-Koopmans Model

Sang Yoon (Tim) Lee

Toulouse School of Economics

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Review from Last Week

- Basic elements of the Solow Growth Model: capital accumulation, population and technology growth
- Neoclassical production function allows one to analyze time path of growth
- Steady state equation allows for cross-country comparisons
- Human capital model incorporates empirical measure for “education”
This Week: Ramsey-Cass-Koopmans Model

- Endogenize physical capital accumulation
- Next week: endogenize technology growth - “ideas”
Representative Agents

- Representative firm uses APF in Solow
- Representative household owns the firm and solves

\[
\max_{c,K} U(c) = \int_0^T e^{-\rho t} u(c_t) dt
\]

subject to

\[
C_t + \dot{K}_t = F(K_t, Z_t L_t) - \delta K_t,
\]

\[
K_0 > 0 \text{ given, } \quad K_T = 0.
\]

where \(c_t \equiv C_t / L_t\), i.e., consumption per capita.

- Assume CRRA utility:

\[
u(c) = \frac{c^{1-\gamma}}{1-\gamma}
\]
As before, normalize stuff by ZL:

\[
\max_{\hat{c}_t, \hat{k}_t} U = \int_0^T e^{-\hat{\rho} t} u(\hat{c}_t) dt
\]

s.t. \( \hat{c}_t + \hat{k}_t = f(\hat{k}_t) - (n + g + \delta) \hat{k}_t, \)

\( \hat{k}_0 > 0 \) given, \( \hat{k}_T = 0. \)

"Effective" discount rate \( \hat{\rho} = \rho - g(1 - \gamma). \)

For discounting to make sense (and for existence of a BGP), we need to assume

\[
\hat{\rho} > 0 \iff \rho > g(1 - \gamma)
\]
Hamiltonian

- Use a (current value) Hamiltonian to describe optimal evolution of \( (\hat{c}_t, \hat{k}_t) \) as functions of time, given \( \hat{k}_0 > 0 \):

\[
\mathcal{H} \left( \hat{c}_t, \hat{k}_t, \lambda_t \right) = u(\hat{c}_t) + \lambda_t \left[ f(\hat{k}_t) - (n + g + \delta)\hat{k}_t - \hat{c}_t \right]
\]

- Sufficient conditions for optimality (similar to f.o.c.'s for a Lagrangian):

\[
\begin{align*}
\hat{c}_t & : \quad \frac{\partial \mathcal{H}}{\partial \hat{c}_t} = 0 \quad \Rightarrow \quad u'(\hat{c}_t) = \lambda_t, \quad (1a) \\
\hat{k}_t & : \quad \frac{\partial \mathcal{H}}{\partial \hat{k}_t} = -\dot{\lambda}_t + \hat{\rho}\lambda_t \quad \Rightarrow \quad -\dot{\lambda}_t = \lambda_t [f'(\hat{k}_t) - (n + g + \delta + \hat{\rho})], \quad (1b) \\
\lambda_t & : \quad \frac{\partial \mathcal{H}}{\partial \lambda_t} = \hat{k}_t \quad \Rightarrow \quad \dot{\hat{k}}_t = f(\hat{k}_t) - (n + g + \delta)\hat{k}_t - \hat{c}_t, \quad (1c)
\end{align*}
\]

along with boundary conditions that depend on \( (T, \hat{k}_T) \).
Continuous Time Optimization

- \((\hat{c}_t, \hat{k}_t, \lambda_t)\): (control, state, costate≈Lagrangian multiplier)
- Want to set \(T = \infty\)
- Inada Conditions:
  \[
  \lim_{c \to 0} u'(c) = \infty, \quad \lim_{c \to \infty} u'(c) = 0 \quad \text{and} \quad \lim_{k \to 0} f'(k) = \infty, \quad \lim_{c \to \infty} f'(k) = 0.
  \]
  CRRA utility and Cobb-Douglas APF satisfy these by assumption
- Boundary Conditions: need as many as there are state-costates!
  \[
  \hat{k}_0 > 0 \quad \text{given,} \quad \lim_{T \to \infty} e^{-\hat{\rho}T}u'(\hat{c}_T)\hat{k}_T = 0.
  \]
- See notes for how to derive the optimality conditions
Balanced Growth Path

• The BGP is simply when \((\lambda_t, \hat{k}_t) = 0:\)

\[
f'(\hat{k}^*) = n + g + \delta + \hat{\rho}
\]

\[
\hat{c}^* = f(\hat{k}^*) - (n + g + \delta)\hat{k}^*.
\]

• The \(\hat{k}^*\) solution to the RCK model is called the “modified golden rule,” as opposed to the “golden rule” \(\hat{k}_S\) with \(s = \alpha:\)

\[
\hat{k}_S = \left(\frac{\alpha}{n + g + \delta}\right)^{\frac{1}{1-\alpha}}
\]

• Compare modified golden rule \(\hat{k}^*\) to the golden rule \(\hat{k}_S\) (homework)
Phase Diagram

- Characterize the dynamics of the system on the $(\hat{c}, \hat{k})$ plane

**c-loci** From (1b), and the strict concavity of the utility function, we know that when

$$f'(\hat{k}) = (n + g + \delta + \hat{\rho})$$

$\hat{c}$ will not change. Note that

1. This is just a straight line parallel to the $c$-axis, such that $\hat{k} = \hat{k}^*$.  
2. To the left of the loci $\lambda$ must decrease, meaning $\hat{c}$ must increase.  
3. Conversely, to the right, $\hat{c}$ must decrease.

**k-loci** From (1c), we know that when

$$\hat{c}(\hat{k}) = f(\hat{k}) - (n + g + \delta)\hat{k}$$

$\hat{k}$ will not change. There are several things to notice:

1. The loci begins at the point $(0, 0)$.  
2. There exist $\hat{k}^* < \hat{k}_S < \hat{k}_u < \infty$ s.t. $\hat{c}$ peaks at $k_S$, and $\hat{c}(\hat{k}_u) = 0$.  
3. Below the loci, $\hat{k}$ is increasing, and above, decreasing.
Saddle Path Uniqueness

- Any optimal path converges to the steady state (BGP).
  - Either of ($\hat{c}, \hat{k}$) cannot hit zero in finite time, since it will violate Inada condition (zero consumption)
  - Must still rule out ($\hat{c}, \hat{k}$) *asymptoting* to zero
- The optimal path is unique and asymptotes to the BGP:
  - Utility and technology are concave, and choice set is convex
  - In continuous time, never actually reach boundary in finite time
Ruling Out $\hat{k} \to 0$

- Can only happen if $\hat{k}$ slows down (strictly) as it approaches 0
- But taking time derivative of (1c),

$$\lim_{\hat{k} \to 0} \frac{\ddot{\hat{k}}}{\hat{k} \to 0} = \lim_{\hat{k} \to 0} \left\{ \left[ f'(\hat{k}) - (n + g + \delta) \right] \dot{\hat{k}} - \dot{\hat{c}} \right\} < 0,$$

- Since $\hat{k} < \hat{k}_S$, $\dot{\hat{k}} < 0$, and $\dot{\hat{c}} > 0$
- Since $\dot{\hat{k}} < 0$, $\hat{k}$ decelerates at a increasing pace

$\Rightarrow$ Asymptoting toward $\hat{k} = 0$ can’t happen.
Ruling Out $\hat{c} \to 0$

- All feasible paths dominated by a path s.t. $\hat{k} \to \hat{k}_u$
- To not violate the TVC, it must be that marginal utility $(\lambda = u'(\hat{c}_t))$ grows slower than $\hat{\rho}$
- From (1b), growth rate of MU as it approaches $\hat{k}_u$ is

$$\lim_{\hat{k} \to \hat{k}_u} \frac{\dot{\lambda}_t}{\lambda_t} = \lim_{\hat{k} \to \hat{k}_u} [-f'(\hat{k}) + n + g + \delta + \hat{\rho}] > \hat{\rho}$$

$\Rightarrow$ TVC is violated, $\hat{c} = 0$ can’t happen.
Local Dynamics I

- Take logs on both sides of (1a) and differentiate w.r.t time:

\[
\frac{u''(\hat{c}_t)}{u'(\hat{c}_t)} \cdot \hat{c}_t = \frac{\lambda_t}{\lambda_t} = - [f'(\hat{k}_t) - (n + g + \delta + \rho)],
\]

With CRRA utility:\(^1\)

\[
\hat{c}_t = \frac{1}{\gamma} [f'(\hat{k}_t) - (n + g + \delta + \rho)] \hat{c}_t \quad (2a)
\]

\[
\hat{k}_t = f(\hat{k}_t) - (n + g + \delta)\hat{k}_t - \hat{c}_t. \quad (2b)
\]

\(^1\)Note that we didn’t need to use the CRRA assumption up to now, except when deriving normalized consumption and the effective discount rate. We haven’t used at all that \(F(\cdot)\) has to be Cobb-Douglas either.
Local Dynamics II

• Taylor approximate the system (2) around the BGP:

\[
\begin{pmatrix}
\dot{\hat{c}}_t \\
\dot{\hat{k}}_t 
\end{pmatrix}
\approx
\begin{pmatrix}
0 & -\sigma \\
-1 & \hat{\rho}
\end{pmatrix}
\begin{pmatrix}
\hat{c}_t - \hat{c}^* \\
\hat{k}_t - \hat{k}^*
\end{pmatrix}
\]

where \( \sigma = -\frac{1}{\gamma} f''(k^*)c^* > 0 \)

• Differentiate the second equation w.r.t. time and substitute back in \((\dot{\hat{c}}_t, \dot{\hat{k}}_t)\) to get:

\[
\ddot{\hat{k}}_t \approx -\dot{\hat{c}}_t + \hat{\rho}\dot{\hat{k}}_t = \sigma(\hat{k}_t - \hat{k}^*) + \hat{\rho}\dot{\hat{k}}_t
\]

or

\[
\ddot{\hat{k}}_t - \hat{\rho}\dot{\hat{k}}_t - \sigma \hat{k}_t \approx -\sigma\hat{k}^*.
\]
Solve 2nd-order ODE

- Solving the second order ODE: get the general and particular solution s.t.

  \[ \hat{k}_t \approx \hat{k}_t^g + \hat{k}_t^p. \]

  Clearly \( \hat{k}_t^p = k^* \).

- The general solution is given by \( \hat{k}_t^g \approx C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t} \) where \( (\alpha_1, \alpha_2) \) are the solutions to \( \alpha^2 - \hat{\rho} \alpha - \beta = 0 \), i.e.

  \[ \alpha_{1,2} = \frac{\hat{\rho} \pm \sqrt{\hat{\rho}^2 + 4\sigma}}{2} = \frac{\hat{\rho}}{2} \pm \sqrt{\frac{\sigma + \frac{\hat{\rho}^2}{4}}}. \]

- The two roots are real and distinct (one larger and one smaller than 0). Let \( \alpha_1 < 0 < \alpha_2 \).
Saddle-Path Stability

• If $C_2 \neq 0$, $\hat{k}_t$ will explode, violating the TVC

• So locally around the BGP, the dynamic system is stable:

$$\hat{k}_t \approx \hat{k}^* + C_1 e^{\alpha_1 t}:$$

and $C_1$ is pinned down by the initial condition $\hat{k}_0$:

$$\hat{k}_t \approx \hat{k}^* + (\hat{k}_0 - \hat{k}^*) e^{\alpha_1 t},$$

• Corresponding solution for $\hat{c}_t$ found from

$$\dot{\hat{k}}_t \approx - (\hat{c}_t - \hat{c}^*) + \hat{\rho}(\hat{k}_t - \hat{k}^*)$$

$$\Rightarrow \hat{c}_t \approx \hat{c}^* + \hat{\rho}(\hat{k}_t - \hat{k}^*) - \hat{k}_t$$

$$\approx \hat{c}^* + (\hat{\rho} - \alpha_1)(\hat{k}_0 - \hat{k}^*) e^{\alpha_1 t}.$$
Speed of Convergence

• \(|\alpha_1|\): “Speed of convergence"

• Increasing in the EIS \(1/\gamma\)
  \(\Rightarrow\) the more the people are willing to substitute toward the future they accumulate faster

• Decreasing in the effective discount rate \(\hat{\rho}\)
  \(\Rightarrow\) the more they discount they accumulate slower.

• Compare with Solow model speed of convergence with golden rule? (homework)
Comparative Statics and Dynamics

• Sudden, temporary increase in $g$?
• Sudden, permanent increase in $g$?
• Temporary increase in $g$ announced $T$ periods ahead?
• Permanent increase in $g$ announced $T$ periods ahead?
Next Week

- Romer’s endogenous growth model