Ramsey-Cass-Koopmans Model

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Review from Last Week

- Basic elements of the Solow Growth Model:
  capital accumulation, population and technology growth
- Neoclassical production function allows one to analyze time path of growth
- Steady state equation allows for cross-country comparisons
- Human capital model incorporates empirical measure for "education"

This Week: Ramsey-Cass-Koopmans Model

- Endogenize physical capital accumulation
- Next week: endogenize technology growth - "ideas"

Representative Agents

- Representative firm uses APF in Solow
- Representative household owns the firm and solves

$$\max_{c,K} U(c) = \int_0^T e^{-\rho t} u(c_t) dt$$
$$\text{s.t.} \quad C_t + \dot{K}_t = F(K_t, Z_t L_t) - \delta K_t,$$
$$K_0 > 0 \text{ given}, \quad K_T = 0.$$

where $c_t \equiv C_t / L_t$, i.e., consumption per capita.

- Assume CRRA utility:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

Normalize!

- As before, normalize stuff by ZL:

$$\max_{\hat{c}_t, \hat{k}_t} U = \int_0^T e^{-\hat{\rho} t} u(\hat{c}_t) dt$$
$$\text{s.t.} \quad \hat{c}_t + \dot{\hat{k}}_t = f(\hat{k}_t) - (n + g + \delta)\hat{k}_t,$$
$$\hat{k}_0 > 0 \text{ given}, \quad \hat{k}_T = 0.$$

- "Effective" discount rate $\hat{\rho} = \rho - g(1 - \gamma)$.
- For discounting to make sense (and for existence of a BGP), we need to assume

$$\hat{\rho} > 0 \quad \Leftrightarrow \quad \rho > g(1 - \gamma)$$

Hamiltonian

- Use a (current value) Hamiltonian to describe optimal evolution of $(\hat{c}_t, \hat{k}_t)$ as *functions of time*, given $\hat{k}_0 > 0$:

$$\mathcal{H}\left(\hat{c}_t, \hat{k}_t, \lambda_t\right) = u(\hat{c}_t) + \lambda_t\left[f(\hat{k}_t) - (n+g+\delta)\hat{k}_t - \hat{c}_t\right]$$

- Sufficient conditions for optimality (similar to f.o.c.'s for a Lagrangian):

$$\hat{c}_t: \quad \frac{\partial \mathcal{H}}{\partial \hat{c}_t} = 0 \quad \Rightarrow \quad u'(\hat{c}_t) = \lambda_t, \tag{1a}$$

$$\hat{k}_t: \quad \frac{\partial \mathcal{H}}{\partial \hat{k}_t} = -\dot{\lambda}_t + \hat{\rho}\lambda_t \quad \Rightarrow \quad -\dot{\lambda}_t = \lambda_t[f'(\hat{k}_t) - (n+g+\delta+\hat{\rho})], \tag{1b}$$

$$\lambda_t: \quad \frac{\partial \mathcal{H}}{\partial \lambda_t} = \dot{\hat{k}}_t \quad \Rightarrow \quad \dot{\hat{k}}_t = f(\hat{k}_t) - (n+g+\delta)\hat{k}_t - \hat{c}_t, \tag{1c}$$

along with boundary conditions that depend on $(T, \hat{k}_T)$.

Continuous Time Optimization

- $(\hat{c}_t, \hat{k}_t, \lambda_t)$: (control, state, costate≈Lagrangian multiplier)
- Want to set $T = \infty$
- Inada Conditions:

$$\lim_{c \to 0} u'(c) = \infty, \qquad \lim_{c \to \infty} u'(c) = 0 \qquad \text{and}$$
$$\lim_{k \to 0} f'(k) = \infty, \qquad \lim_{c \to \infty} f'(k) = 0.$$

  CRRA utility and Cobb-Douglas APF satisfy these by assumption

- Boundary Conditions: need as many as there are state-costates!

$$\hat{k}_0 > 0 \quad \text{given,} \qquad \lim_{T \to \infty} e^{-\hat{\rho} T} u'(\hat{c}_T) \hat{k}_T = 0.$$

- See notes for how to derive the optimality conditions

Balanced Growth Path

- The BGP is simply when $(\dot{\lambda}_t, \dot{\hat{k}}_t) = 0$:

$$f'(\hat{k}^*) = n + g + \delta + \hat{\rho}$$
$$\hat{c}^* = f(\hat{k}^*) - (n + g + \delta)\hat{k}^*.$$

- The $\hat{k}^*$solution to the RCK model is called the "modified golden rule," as opposed to the the "golden rule" $\hat{k}_S$ with $s = \alpha$:

$$\hat{k}_S = \left(\frac{\alpha}{n + g + \delta}\right)^{\frac{1}{1-\alpha}}$$

- Compare modified golden rule $\hat{k}^*$ to the golden rule $\hat{k}_S$ (homework)

Phase Diagram

- Characterize the dynamics of the system on the $(\hat{c}, \hat{k})$ plane

c-loci From (1b), and the strict concavity of the utility function, we know that when

$$f'(\hat{k}) = (n + g + \delta + \hat{\rho})$$

$\hat{c}$ will not change. Note that

1. This is just a straight line parallel to the c-axis, such that $\hat{k} = \hat{k}^*$.
2. To the left of the loci λ must decrease, meaning $\hat{c}$ must increase.
3. Conversely, to the right, $\hat{c}$ must decrease.

k-loci From (1c), we know that when

$$\hat{c}(\hat{k}) = f(\hat{k}) - (n + g + \delta)\hat{k}$$

$\hat{k}$ will not change. There are several things to notice:

1. The loci begins at the point $(0, 0)$.
2. There exist $\hat{k}^* < \hat{k}_S < \hat{k}_u < \infty$ s.t. $\hat{c}$ peaks at k_S, and $\hat{c}(\hat{k}_u) = 0$.
3. Below the loci, $\hat{k}$ is increasing, and above, decreasing.

Saddle Path Uniqueness

- Any optimal path converges to the steady state (BGP).
  - Either of $(\hat{c}, \hat{k})$ cannot hit zero in finite time, since it will violate Inada condition (zero consumption)
  - Must still rule out $(\hat{c}, \hat{k})$ *asymptoting* to zero
- The optimal path is unique and asymptotes to the BGP:
  - Utility and technology are concave, and choice set is convex
  - In continuous time, never actually reach boundary in finite time

Ruling Out $\hat{k} \to 0$

- Can only happen if $\hat{k}$ slows down (strictly) as it approaches 0
- But taking time derivative of (1c),

$$\lim_{\hat{k} \to 0} \ddot{\hat{k}} = \lim_{\hat{k} \to 0} \left\{ [f'(\hat{k}) - (n + g + \delta)]\dot{\hat{k}} - \dot{\hat{c}} \right\} < 0,$$

- Since $\hat{k} < \hat{k}_S$, $\dot{\hat{k}} < 0$, and $\dot{\hat{c}} > 0$
- Since $\ddot{\hat{k}} < 0$, $\hat{k}$ decelerates at a increasing pace

$\Rightarrow$ **Asymptoting toward $\hat{k} = 0$ can't happen.**

Ruling Out $\hat{c} \to 0$

- All feasible paths dominated by a path s.t. $\hat{k} \to \hat{k}_u$
- To not violate the TVC, it must be that marginal utility ($\lambda = u'(\hat{c}_t)$) grows slower than $\hat{\rho}$
- From (1b), growth rate of MU as it approaches $\hat{k}_u$ is

$$\lim_{\hat{k} \to \hat{k}_u} \frac{\dot{\lambda}_t}{\lambda_t} = \lim_{\hat{k} \to \hat{k}_u} [-f'(\hat{k}) + n + g + \delta + \hat{\rho}] > \hat{\rho}$$

$\Rightarrow$ **TVC is violated, $\hat{c} = 0$ can't happen.**

Local Dynamics I

- Take logs on both sides of (1a) and differentiate w.r.t time:

$$\frac{u''(\hat{c}_t)}{u'(\hat{c}_t)} \cdot \dot{\hat{c}}_t = \frac{\dot{\lambda}_t}{\lambda_t} = -[f'(\hat{k}_t) - (n + g + \delta + \hat{\rho})],$$

With CRRA utility:[1]

$$\dot{\hat{c}}_t = \frac{1}{\gamma}[f'(\hat{k}_t) - (n + g + \delta + \hat{\rho})]\hat{c}_t \tag{2a}$$

$$\dot{\hat{k}}_t = f(\hat{k}_t) - (n + g + \delta)\hat{k}_t - \hat{c}_t. \tag{2b}$$

[1]Note that we didn't need to use the CRRA assumption up to now, except when deriving normalized consumption and the effective discount rate. We haven't used at all that $F(\cdot)$ has to be Cobb-Douglas either.

Local Dynamics II

- Taylor approximate the system (2) around the BGP:

$$\begin{pmatrix} \dot{\hat{c}}_t \\ \dot{\hat{k}}_t \end{pmatrix} \approx \begin{pmatrix} 0 & -\sigma \\ -1 & \hat{\rho} \end{pmatrix} \begin{pmatrix} \hat{c}_t - \hat{c}^* \\ \hat{k}_t - \hat{k}^* \end{pmatrix}$$

  where $\sigma = -\frac{1}{\gamma} f''(k^*) c^* > 0$

- Differentiate the second equation w.r.t. time and substitute back in $(\dot{\hat{c}}_t, \dot{\hat{k}}_t)$ to get:

$$\ddot{\hat{k}}_t \approx -\dot{\hat{c}}_t + \hat{\rho}\dot{\hat{k}}_t = \sigma(\hat{k}_t - \hat{k}^*) + \hat{\rho}\dot{\hat{k}}_t$$

  or

$$\ddot{\hat{k}}_t - \hat{\rho}\dot{\hat{k}}_t - \sigma k_t \approx -\sigma \hat{k}^*.$$

Solve 2nd-order ODE

- Solving the second order ODE: get the general and particular solution s.t.

$$\hat{k}_t \approx \hat{k}_t^g + \hat{k}_t^p.$$

 Clearly $\hat{k}_t^p = k^*$.

- The general solution is given by $\hat{k}_t^g \approx C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t}$ where (α_1, α_2) are the solutions to $\alpha^2 - \hat{\rho}\alpha - \beta = 0$, i.e.

$$\alpha_{1,2} = \frac{\hat{\rho} \pm \sqrt{\hat{\rho}^2 + 4\sigma}}{2} = \frac{\hat{\rho}}{2} \pm \sqrt{\sigma + \frac{\hat{\rho}^2}{4}}.$$

- The two roots are real and distinct (one larger and one smaller than 0). Let $\alpha_1 < 0 < \alpha_2$.

Saddle-Path Stability

- If $C_2 \neq 0$, $\hat{k}_t$ will explode, violating the TVC
- So locally around the BGP, the dynamic system is stable:

$$\hat{k}_t \approx \hat{k}^* + C_1 e^{\alpha_1 t} :$$

and C_1 is pinned down by the initial condition $\hat{k}_0$:

$$\hat{k}_t \approx \hat{k}^* + (\hat{k}_0 - \hat{k}^*) e^{\alpha_1 t},$$

- Corresponding solution for $\hat{c}_t$ found from

$$\begin{aligned} & \dot{\hat{k}}_t \approx -(\hat{c}_t - \hat{c}^*) + \hat{\rho}(\hat{k}_t - \hat{k}^*) \\ \Rightarrow \qquad & \hat{c}_t \approx \hat{c}^* + \hat{\rho}(\hat{k}_t - \hat{k}^*) - \dot{\hat{k}}_t \\ & \quad \approx \hat{c}^* + (\hat{\rho} - \alpha_1)(\hat{k}_0 - \hat{k}^*) e^{\alpha_1 t}. \end{aligned}$$

Speed of Convergence

- $|\alpha_1|$: "Speed of convergence"
- Increasing in the EIS $1/\gamma$

  $\Rightarrow$ the more the people are willing to substitute toward the future they accumulate faster
- Decreasing in the effective discount rate $\hat{\rho}$

  $\Rightarrow$ the more they discount they accumulate slower.
- Compare with Solow model speed of convergence with golden rule? (homework)

Comparative Statics and Dynamics

- Sudden, temporary increase in g?
- Sudden, permanent increase in g?
- Temporary increase in g announced T periods ahead?
- Permanent increase in g announced T periods ahead?

Next Week

- Romer's endogenous growth model