Solow Growth Model

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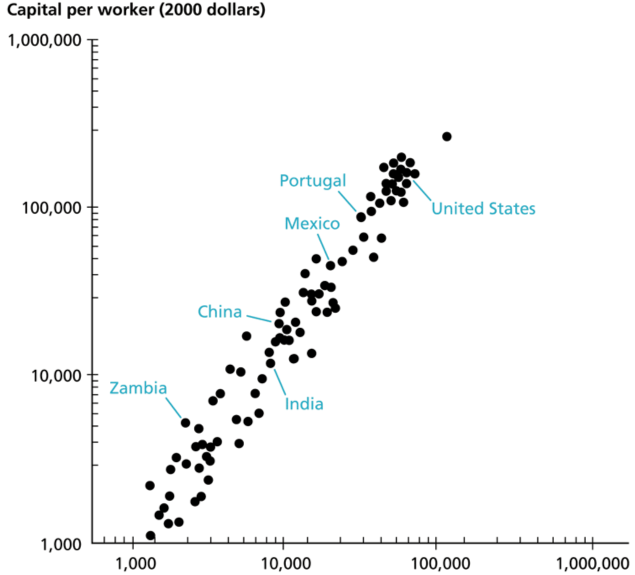
This Week: Industrialized Countries

Kaldor Facts: since we ever measured such things,

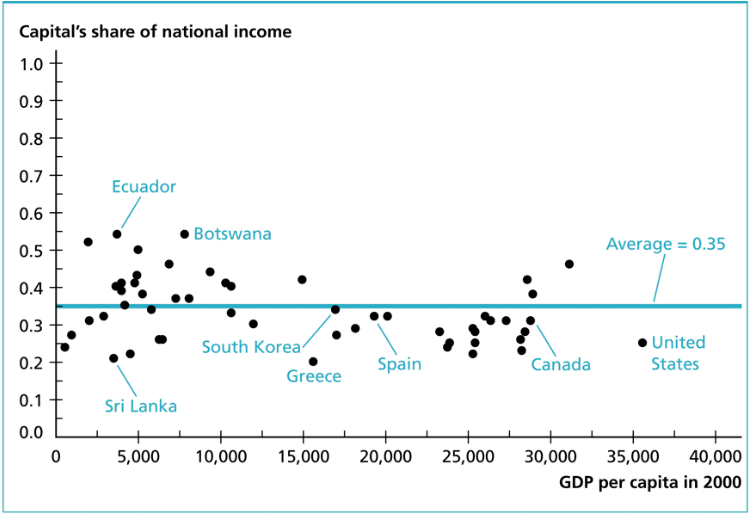
1. capital per worker grows at nearly constant rate
   $\Rightarrow$ *also constant across countries' GDP per worker in log-scale!*
2. capital-output ratio is nearly constant over time
3. capital-labor income shares are nearly constant over time
   $\Rightarrow$ *also constant across countries!*
4. return to capital (interest rate) is nearly constant over time

We want to a model that can explain these facts: keep that in mind, we'll validate the Solow model in terms of these facts

GDP and Capital per Worker



Capital Share of Income



Source: Bernanke and Gürkaynak (2002), table 10 and note 18.

What's a Model?

- Important to realize "models" are just an approximation
  (Newtonian mechanics is wrong, but still a good approximation even today)
- constructed to explain a specific (set of) fact(s)
- will always be wrong in other dimensions - this goes for any model, even physics or mathematical models
  (Newtonian mechanics can't explain electromagnetic fields)

Exogenous variables	$\Rightarrow$	**Model**	$\Rightarrow$	**Endogenous variables**
mass, acceleration	$\Rightarrow$	Newton	$\Rightarrow$	force, position
savings, population	$\Rightarrow$	Solow	$\Rightarrow$	GDP&Capital/capita growth

What's a *Good* Model?

- Strength of a model is
  - Does it explain the facts it's trying to explain?

    (Newton explained well the movements of planets and everything down to the atomic level—but not the atoms themselves)
  - Is it clear *why* it can explain what it's trying to explain
  - Does it fail spectacularly everywhere else, or does it also "sort of" explain things it's *not* designed to explain?

    (Newton framework worked well till the late 19th century...)
- If the model doesn't fit, can we change it a bit to "make it" fit?

  (...with some modifications.)
  - If the changes are too much, we are obviously in need of a new model

    (Newtonian mechanics failed with the discovery of X-rays and neutrons, was replaced by general relativity and quantum mechanics. In short, the model is ***wrong****.)*
  - This hasn't really happened yet with the Solow model though...we are still using variants

Solow Growth Model

- Solow growth model is significant because
  - easy to understand
  - can explain Kaldor facts
- Can also empirically explain in a simple way the:
  - growth of a single country (law of motion)
  - cross country growth rate comparisons (at the steady state)
- Just a simple function that takes
  - growth factors as the domain *(savings, population growth)*
  - GDP growth as its range *(also capital growth)*
  - easily modifiable to explain other facts *(human capital, technology, etc.)*

Neoclassical Production Function

Aggregate Production Function (APF)

$$Y = F(K, ZL) = K^{\alpha}(ZL)^{1-\alpha}$$

Y: output

Z: technology (in this case, "labor-augmenting," but not important)

Basically, anything that we can't explain, just call it technology

K: capital

L: labor (population)

α: factor shares

Remember, the APF is significant only because it matches empirical facts...doesn't mean production actually happens this way!

Convenient Characteristics

- Constant Returns to Scale

$$
\begin{aligned}
Y &= (tK)^{\alpha}(tZL)^{1-\alpha} \\
&= tK^{\alpha}(ZL)^{1-\alpha}
\end{aligned}
$$

- Constant factor income shares

$$
\max_{K,L} \{F(K, ZL) - RK - wL\}
$$

f.o.c.

$$
\begin{aligned}
R &= \frac{\alpha Y}{K} && \Rightarrow RK = \alpha Y \\
w &= \frac{(1-\alpha)Y}{L} && \Rightarrow wL = (1-\alpha)Y
\end{aligned}
$$

Desirable Characteristics

- **Kaldor fact 3: factor income shares are constant**

$$\frac{RK}{Y} = \alpha, \qquad \frac{wL}{Y} = 1 - \alpha$$

- **Kaldor fact 2: capital-output ratio is constant**
  $\Rightarrow K/Y$ must be constant
- **Kaldor fact 4: return to capital is constant**
  $\Rightarrow$ but if K/Y is constant, so is R.
  $\Rightarrow$ i.e., Kaldor fact 4 becomes a subset of fact 2

Fact 3 is done, and if we get fact 2, we automatically have fact 4.
Still need to explain facts 1 and 2.

Exogenous and Endogenous Variables

Key to understanding models!

- **Exogenous Variables**: things that don't change, we just assume that they are constant, or mechanically changing from *outside the model*
- **Endogenous Variables**: things we have to solve for *within the model*
- **Solving a model** means expressing endogenous variabes in terms of exogenous variables

When setting up a model,

- I will always point out what is exogenous and endogenous
- You must always be aware of what is exogenous and endogeneous

Law of Motion in Discrete Time

- Assume that the aggregate savings rate is exogenous
- You may be used to seeing things like

$$K_{t+1} = sY_t + (1 - \delta)K_t$$
$$K_{t+1} - K_t = sY_t - \delta K_t$$

  where

s exogenous savings rate

δ depreciation rate of capital, also exogenous

- Note that both are *measurable in objective units, i.e. real dollars (euros)*

Law of Motion in Continuous Time

That's what happens in **1** unit of time (for example, 1 year, 1 quarter, 1 month...):

$$\frac{K_{t+1} - K_t}{\mathbf{1}} = sY_t - \delta K_t$$

If Δ units,

$$\frac{K_{t+\Delta} - K_t}{\Delta} = sY_t - \delta K_t$$

So if time is continuous,

$$\frac{dK(t)}{dt} = sY(t) - \delta K(t)$$

A Little More Explanation

$$\frac{dK(t)}{dt} = sY(t) - \delta K(t)$$

- $K(t), Y(t)$: view these quantities as **functions of time**, i.e. the values at time t. These are **endogenous** variables, we have to solve for them. Usually we suppress writing t.
- dt: very short time interval
- dK: change in K in very short time interval

$\Rightarrow$ dK/dt: change in K in 1 unit of time! The **time derivative** of $K(t)$.

Growth Rates in Continuous Time

- For growth and development we consider long enough time intervals (decades, centuries) that even a year can be considered an instant
- Capital evolves as:

$$\frac{dK}{dt} = sY - \delta K$$

  Then, the **growth rate** of capital:

$$\frac{(K_{t+\Delta} - K_t)/\Delta}{K_t} \approx \frac{dK/dt}{K} = \frac{d \log K}{dt} = \frac{sY}{K} - \delta$$

- At this point,

  K, Y endogneous (something to solve for)

  s, δ exogenous (something we would plug in some *measured* numbers for)

More Assumptions

- Now assume that technology and population also grow at constant, **exogenous** rates g and n, respectively:

$$\frac{d\log Z}{dt} = \frac{dZ/dt}{Z} = g$$
$$\frac{d\log L}{dt} = \frac{dL/dt}{L} = n$$

- and that
  - All technology can be applied to production
  - Entire population works for production
- Z, or technology, is everything we cannot explain by observables
- Also called **T**otal **F**actor **P**roductivity (TFP)

Normalization

Now remember...

Kaldor fact 1: capital per worker $\equiv$ K/L grows at a constant rate

So let's divide everything by ZL ("normalization"):

$$\frac{F(K, ZL)}{ZL} \equiv \frac{Y}{ZL} \equiv \hat{y}, \quad \frac{K}{ZL} \equiv \hat{k}$$

so

$$\hat{y} = \frac{K^{\alpha}(ZL)^{1-\alpha}}{ZL} = \hat{k}^{\alpha}$$

Normalized Law of Motion

Take logs on $\hat{k} \equiv K/ZL$:

$$\log \hat{k} = \log K - \log Z - \log L$$

Take time derivatives to get growth rate of normalized capital

$$\frac{d\hat{k}/dt}{\hat{k}} = \underbrace{\frac{dK/dt}{K}}_{\text{we know this}} - \frac{dZ/dt}{Z} - \frac{dL/dt}{L}$$

$$= \underbrace{\frac{sY}{K} - \delta}_{\text{growth rate of aggregate capital}} - g - n$$

Solving the Model

- The normalized law of motion

$$\frac{d\hat{k}}{dt} = s\hat{y} - (n + g + \delta)\hat{k}$$

- But, remember that what we want is the growth rate of $k = K/L$ to be constant (Kaldor Fact 1), let's call it κ:

$$\frac{dk/dt}{k} = \underbrace{\frac{dZ/dt}{Z}}_{g} + \underbrace{\frac{d\hat{k}/dt}{\hat{k}}}_{x} = g + x = \kappa$$

where x is "something" we want to solve for.

Kaldor Fact 1

- If we knew x, we can solve for normalized capital:

$$\hat{k}^* = \frac{s\hat{y}^*}{n + g + \delta + x}$$

- But by assumption, Z is everything we don't know, which has a growth rate of g.
- And we got $\hat{k}$ by dividing k by Z in the first place.
- So set $x = 0$ and suppress everything in g! Then:
    1. K/L grows at a constant rate κ
    2. That constant is $\kappa = g$
    3. We cannot explain *where* the magic number g comes from, but we can explain why the growth rate is constant

Steady State (Balanced Growth Path)

- Steady State: when nothing changes over time (growth rate 0)
- BGP: when things grow at constant rate
- In our case, $\frac{dk/dt}{k} = \kappa \equiv g$ and $\frac{d\hat{k}/dt}{\hat{k}} = 0$. Then

$$\hat{k}^* = \frac{s\hat{y}^*}{n+g+\delta} = \frac{s\hat{k^*}^{\alpha}}{n+g+\delta},$$

so the BGP level of normalized capital and output:

$$\hat{k}^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} \qquad \Rightarrow \qquad \hat{y}^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}.$$

so the BGP is a steady state w.r.t. $\hat{k}$

- but we are at a BGP w.r.t. k (**not** steady state, next slide)

Back to What We Wanted

Kaldor fact 1: capital per worker grows at constant rate:

Clearly, $k \equiv K/L$ grows at constant rate g

$$k^* = Z\hat{k}^* \qquad \Rightarrow \qquad \frac{d\log k^*}{dt} = \underbrace{\frac{d\log Z}{dt}}_{g} + \underbrace{\frac{d\log \hat{k}^*}{dt}}_{0} = g$$

And so does $y \equiv Y/L = Z^{1-\alpha}k^{\alpha}$ (try it).

Kaldor fact 2: capital-output ratio is constant:

$$\frac{K^*}{Y^*} = \frac{k^*}{y^*} = \frac{\hat{k}^*}{\hat{y}^*} = \frac{s}{n+g+\delta}$$

so obviously, return to capital R is constant.

Whether this ratio makes sense, we'll see next week.

Summary

- **Model setup:**
  - "Neoclassical production function": $Y = F(K, ZL) = K^{\alpha}(ZL)^{1-\alpha}$
    This already gives us
    1. Capital income share is constant
    2. Return to capital constant if K/Y constant
    3. Y/L log-linear in K/L
  - Law of motion: $dK/dt = sY - \delta K, \quad dZ/dt = gZ, \quad dL/dt = nL.$
- **Normalize:** $\hat{y} = Y/AL, \quad \hat{k} = K/AL$
- **Solve for:** $d\hat{k}/dt = 0$
- The solution admits
  1. Constant growth of K/L, Y/L
  2. Constant K/Y

Why Economists use Logarithms

- One ubiquitously used Taylor approximation in math: in the vicinity of $x = 0$,

$$\log(1+x) \approx \log(1+0) + \frac{1}{1+0} \cdot (x-0) = x$$

- So suppose
  - y_t: GDP/capita today
  - y_{t+1}: GDP/capita tomorrow

  then

$$\log y_{t+1} - \log y_t = \log\left(\frac{y_{t+1}}{y_t}\right) = \log\left(1 + \frac{y_{t+1} - y_t}{y_t}\right)$$
$$\approx \frac{y_{t+1} - y_t}{y_t} = \frac{\Delta y_t}{y_t},$$

  which is the growth rate, or percentage differences.

- You don't have to know the math, but remember: **log differences gives the (net) growth rate** or **percentage differences**

Time Derivatives in Continuous Time

For most of this course, we will use time derivatives:

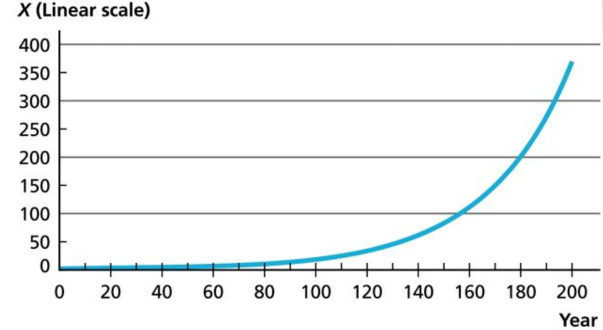
- For *any* variable that changes over time, we can view it as a function of time: e.g. GDP/capita y at time t is $y_t = y(t)$
- If time is continous, we can take the derivative!

$$y'(t) = \frac{dy(t)}{dt}$$

- This helps us get easy expressions for the growth rate

$$\lim_{\Delta \to 0} [\log y_{t+\Delta} - \log y_t] = \frac{d \log y(t)}{dt} = \frac{y'(t)}{y(t)},$$

  which again is the growth rate, or percentage differences.
- You don't have to know the math, but remember:
  taking logs then differentiating gives the (net) growth rate or **percentage differences**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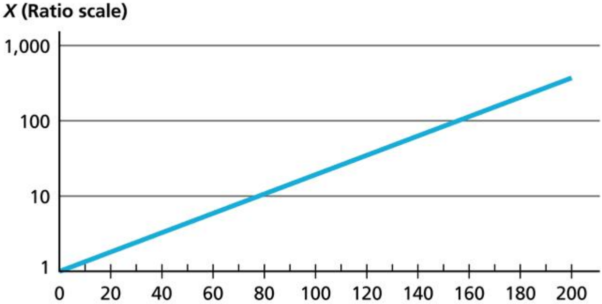
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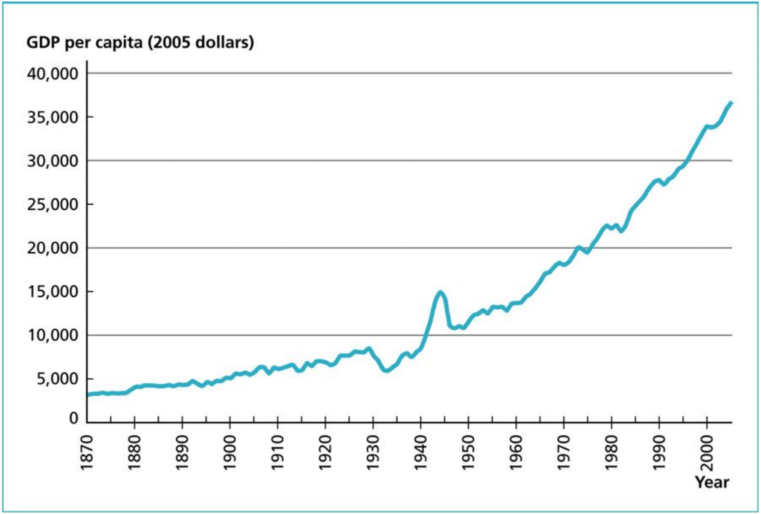
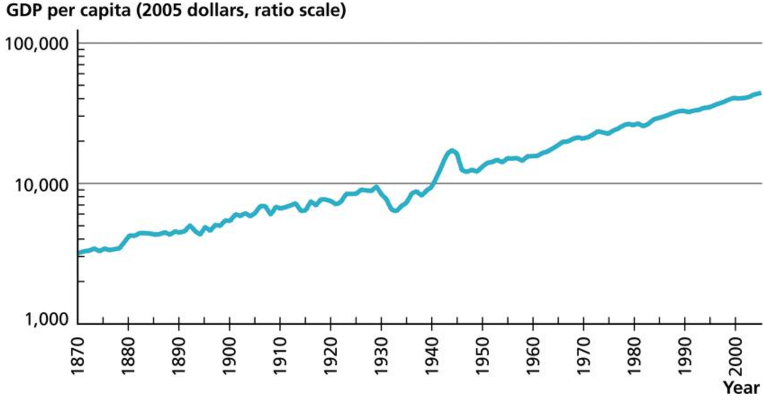
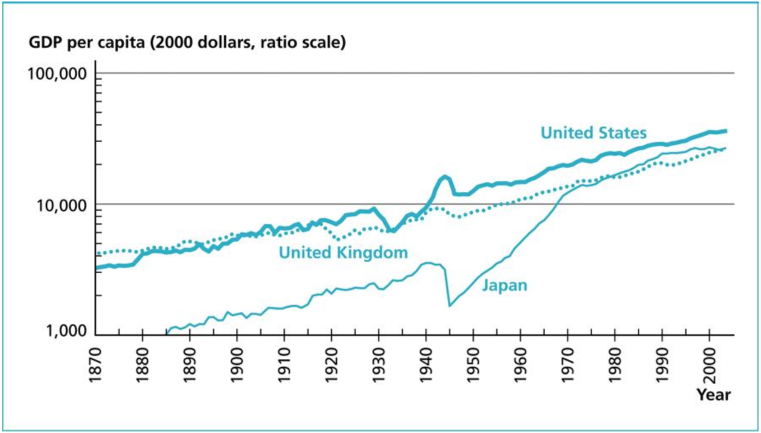
US GDP/capita, 1870-2005



US GDP/capita growth, 1870-2005



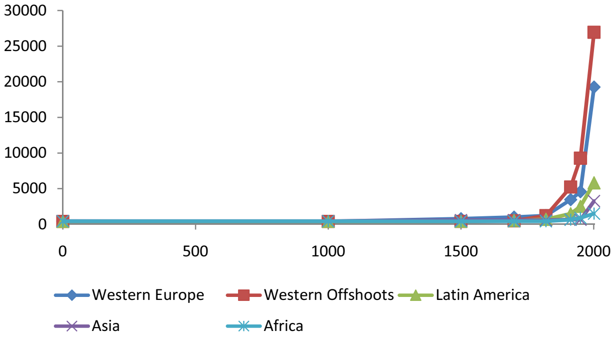
US/UK/Japan growth



Source: Maddison (1995), Heston, Summers, and Aten (2006), World Bank (2007a).

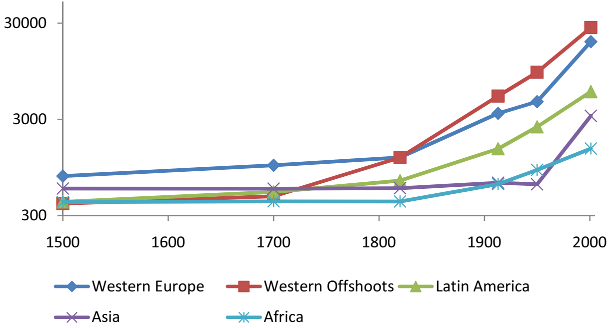
Economic Growth in Historical Perspective

- Income per capita in world regions from 0 to 2000 (Maddison 2003):



Economic Growth in Historical Perspective

- Income per capita in world regions from 1500 to 2000, logarithmic scale (Maddison 2003):



World growth in a Snapshot